CBE 30357
Fall 2017
Test #1
9/28/17

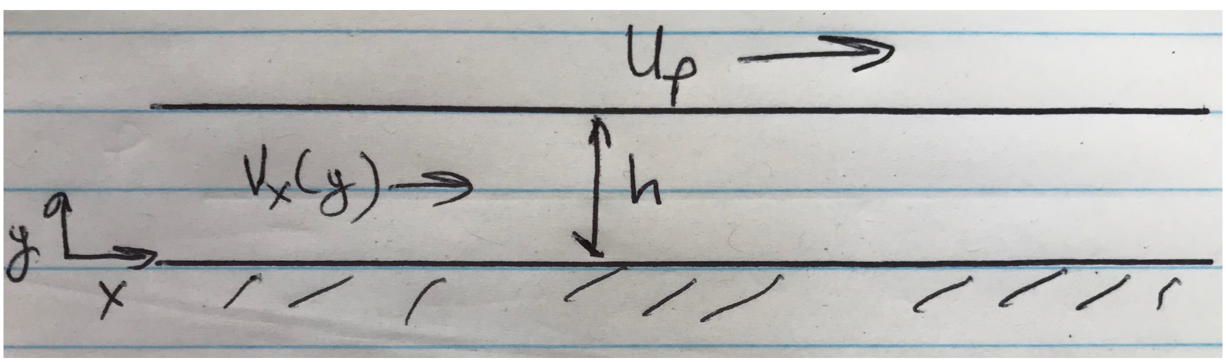
1. Flows in a rectangular geometry. (35 points)

An example of situations where physiological fluids occur in channels with rectangular or square cross sections are inside "microfluidic" devices that are used or being developed for various medical tests. The "micro" part is to make the amount of fluid as small as possible (but it still must be gathered free from contamination) and the automation to raise the reliability and accuracy of the tests. The working sections of microfluidic devices are often disposable plastic slabs about the size of microscope slides with channels for the sample fluid and analytes and openings for electrodes. Fluid is moved around either by a pressure gradient or an electric field gradient.

We will just consider the most simple set of flow scenarios that are motivated by these devices.

Consider the flow in a infinitely wide (in the z direction) rectangular channel as shown here.

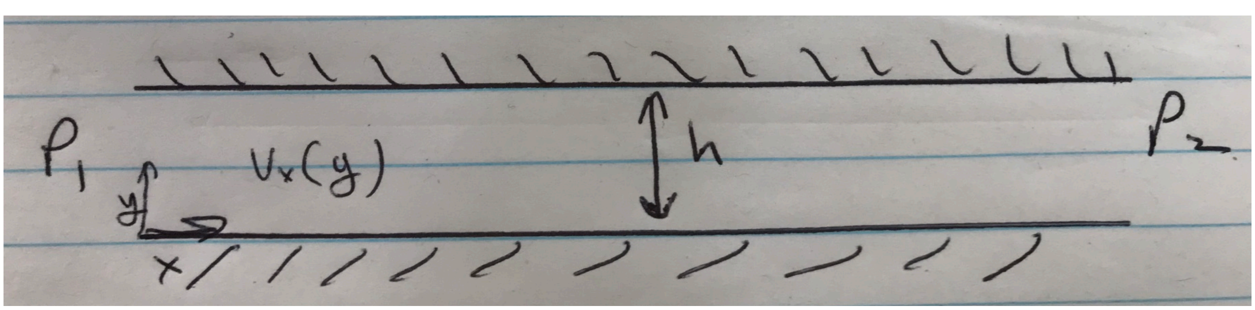
a. Suppose that the flow is driven by a moving top surface. Find the non zero terms of the Navier-Stokes equations, the boundary conditions and use these to get the velocity profile.



b. If the viscosity of the liquid is 0.003 kg/(m-s), h = 0.1 mm and U_p = 3 cm/s, what is the shear stress in the channel?

c. What speed does the plate have to move to cause a flow of 0.05 ml/s per a width (out of plane in the z direction) of 2 mm for a section of channel that is 1.5 cm long.

Now consider the same flow caused by a pressure gradient. (You may wish to locate your axes either on the bottom plate or in the middle of the channel.)



d. What value of P_1-P_2 is required to match the flow rate of part c? You will probably need to choose the correct differential equation for pressure driven flow, integrate it with the correct boundary conditions and then calculate the volumetric flow rate to allow numerical determination of the flow rate.

e. What is the value of the maximum shear stress for the flow conditions of part d.

f. Calculate the power requirements for the pressure driven flow and the moving plate flow for the flow rates of part c. You may recall the power is volumetric flow times pressure change for a pressure driven flow and you can adapt the standard physics expression for work as

$$W = \int \vec{F} \cdot d\vec{l}$$ for the sliding surface.

g. Which flow configuration is more efficient in terms of energy/power use?

2. **Cerebral Spinal Fluid flow in the brain. (50 points)**

You may recall the video clip that showed images of a "fluid front" moving within a mouse brain. The claim in the narration was that this was *cerebral spinal fluid (CSF)* moving through the brain and that the pathways of entry (or flow) were the regions outside of the arteries. Further speculation was that this pathway exists for humans and is enhanced when we are sleeping and so one of the benefits of sleep is this "cleansing" of the brain by this fluid flow around the brain tissue. (It is thought that, one way or another, the circulation of *CSF* within your head is playing the same role as lymph fluid and the lymph circulation system does in the rest of your body.)

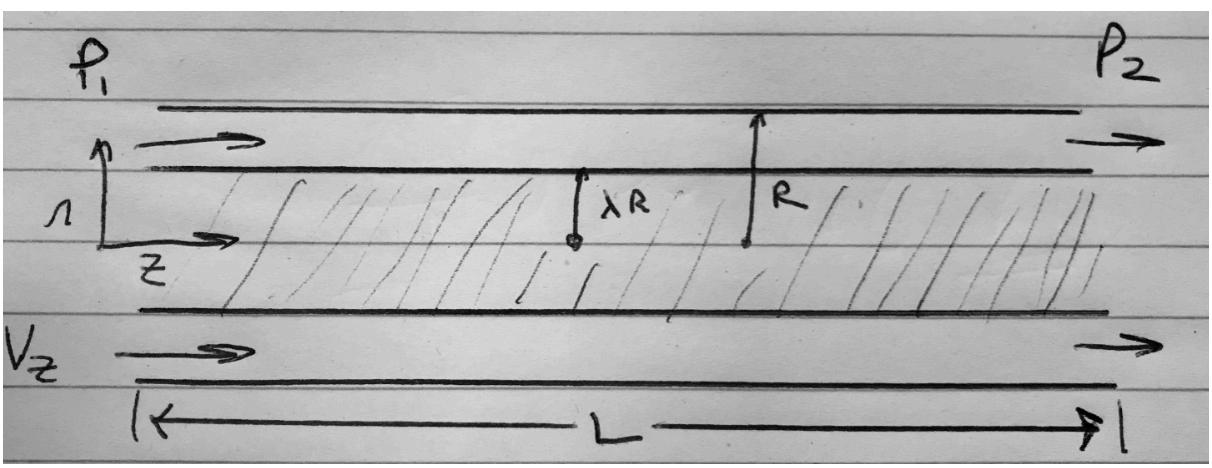
Even more speculation was that limitations in the circulation could be related to the development of *Alzheimer's* disease.

Alzheimer's disease is certainly a problem. Its cause and exact progression are unknown. The approved treatments either don't work at all or delay onset only for a short time. According to the *Wall Street Journal* (9/27/17), about 5.5 million Americans have the disease and this number is expected to rise as the population continues to age. In the article about the failure of a potential new treatment, "intepirdine", the potential market for a new treatment is billions of dollars per year. (Presumably, once you started taking a treatment, you would take it for life!)

We would like to examine the fluid flow aspects of *CSF* flowing along the outside of arteries. Presumably the "gap" available for flow does increase if the average blood pressure inside the arteries is lower because they are elastic.

Let's see what we can figure out about this situation.

The flow geometry is shown here.



The CSF will flow in the annular region (outside of the artery) between $r = \lambda R$ and $r = R$. The flow is pressure driven and for the first part of the problem we will consider only the initial entry into the brain down this one artery that is of length L.

The *CSF* is Newtonian with a viscosity μ, and density ρ.

a. Write down the non-zero terms of the Navier-Stokes equations that govern this flow.
b. What boundary conditions are needed to get an expression for the velocity profile in the annular region?
c. Use the answers to *a* and *b* to get the velocity profile, $v_z(r)$.

To save time, the following result is given.

$$\int_0^R 2\pi r\, v_Z(r)\, dr = \frac{\pi G R^4 \left(\left(\lambda^2 - 1\right)^2 - \left(\lambda^4 - 1\right) \log(\lambda) \right)}{8\mu \log(\lambda)}, \quad G = -dP/dz$$

d. What is being calculated?

A cerebral artery has a radius of 0.3 cm and the according to a paper by Gamble et al. 1994, the pressure strain elastic modulus of an artery is about 60 kPa. If your blood pressure drops by about 30% when you sleep, this means that the "gap" is about 1 mm.

e. For this artery, for a length of 5 cm and an average P_1 - P_2 of 0.01 kPa (=10 kg-m/s^2) , what is the volumetric flow rate of *CSF* if its viscosity is 0.002 kg/(m-s) and its density is 1050 kg/m^3?

We might have occasion to do the following expansions of the result from *d*.

$$\frac{\pi G R^4 \left(\left(\lambda^2 - 1\right)^2 - \left(\lambda^4 - 1\right) \log(\lambda) \right)}{8\mu \log(\lambda)} \approx$$

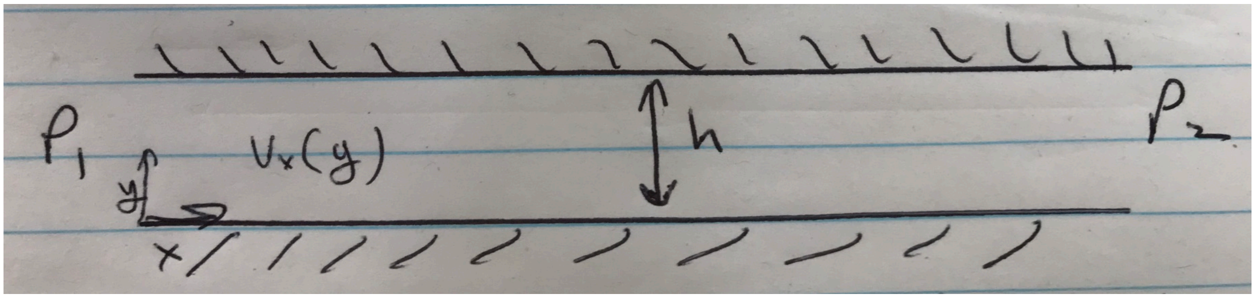
$$-\frac{(\lambda - 1)^3 \left(G R^4\right)}{12\mu} - \frac{(\lambda - 1)^4 \left(G R^4\right)}{24\mu} + O\left((\lambda - 1)^5\right),$$

$$\frac{\pi G R^4 \left(\left(\lambda^2 - 1\right)^2 - \left(\lambda^4 - 1\right) \log(\lambda) \right)}{8\mu \log(\lambda)} \approx$$

$$\frac{\pi G R^4 (\log(\lambda) + 1)}{8\mu \log(\lambda)} - \frac{\lambda^2 \left(\pi G R^4\right)}{4(\mu \log(\lambda))} - \frac{\lambda^4 \left(\pi G R^4 (\log(\lambda) - 1)\right)}{8(\mu \log(\lambda))} + O\left(\lambda^5\right),$$

$$\frac{\pi G R^4 \left(\left(\lambda^2 - 1\right)^2 - \left(\lambda^4 - 1\right) \log(\lambda) \right)}{8\mu \log(\lambda)} \approx \frac{\pi G h^3 R}{6\mu} - \frac{(\pi G) h^4}{12\mu} + O\left(h^5\right), \quad h = R - \lambda R.$$

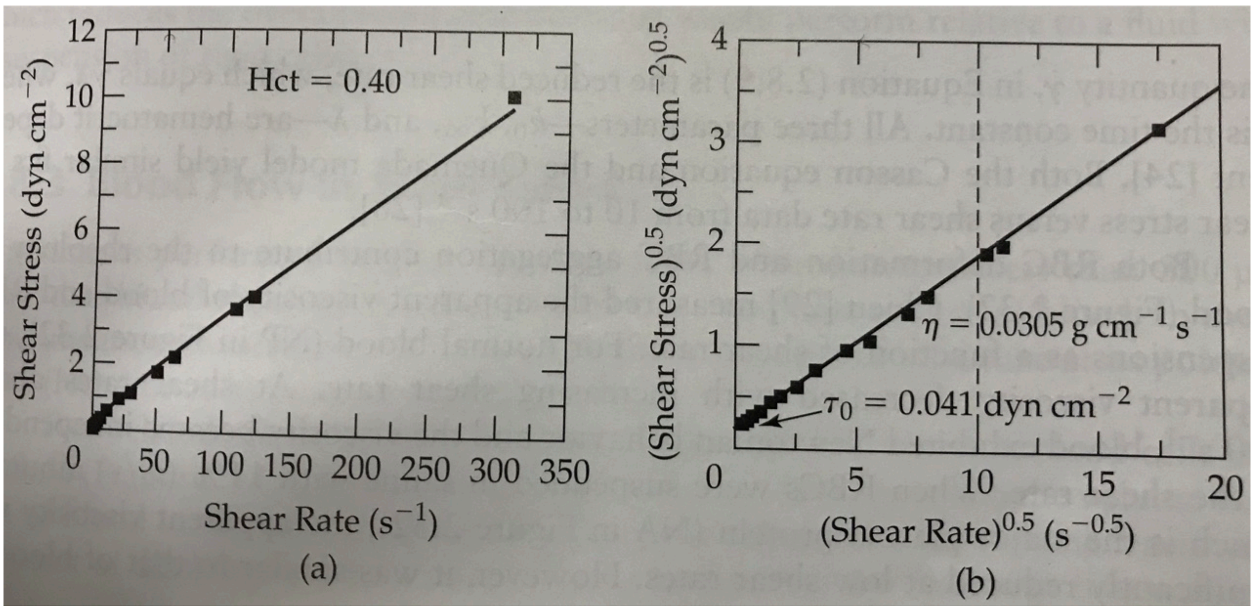
f. Show that this result, in the appropriate limit, matches the flow in an infinitely wide rectangular channel that derived in problem 1.



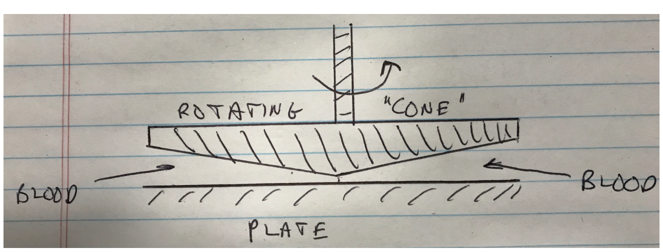
3. Viscosity of Blood and resulting flow (15 points)

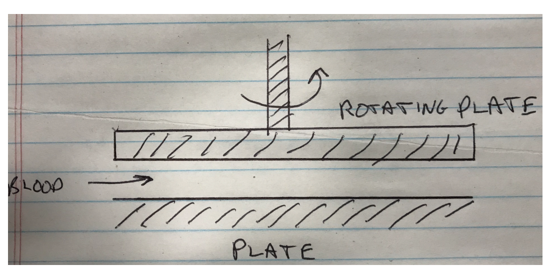
The shear stress τ_{xy},(or τ_{rz} or $\tau_{r\theta}$) — shear rate $\dot{\gamma}$ behavior for human blood is shown in the two figures below. Blood exhibits both a "yield stress", τ_0 consistent with a Bingham plastic and some shear thinning at low shear rates.

The selection of the axes for the figure on the right is motivated by the form of the "Casson equation, $\sqrt{\tau_{yx}} = \sqrt{\eta\dot{\gamma}} + \sqrt{\tau_0}$, that is considered to be a good constitutive model for blood flow behavior.



a. What average velocity would occur in a 1 mm high (h = 1 mm) channel that is really wide if the pressure drop is 10 kg-m/s^2 over a distance of 1 cm. <u>An approximate answer is appropriate for this problem</u>, but explain why you can predict how good the answer will be.
b. A "cone and plate viscometer" was used to measure the shear stress — shear rate relation for a blood. For a non-Newtonian fluid what advantage would this geometry have compared to a parallel plate rotating flow field?





Formulae:

Surface Tension pressure jump across a curved interface:

$$\Delta p = \frac{2\gamma}{R}$$

$$\text{Area for circle} = \int_0^{2\pi}\int_0^R r\,dr\,d\theta$$

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}\right)$

TABLE 3.2

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \tag{3.3.18a}$$

y component

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \tag{3.3.18b}$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] \tag{3.3.18c}$$

Cylindrical coordinates

r component

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{zr}}{\partial z}\right] \tag{3.3.19a}$$

θ component

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] = \rho g_\theta - \frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right] \tag{3.3.19b}$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] \tag{3.3.19c}$$

Spherical coordinates

r component

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] \tag{3.3.20a}$$

TABLE 3.3

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \frac{\partial \mathrm{v}_x}{\partial x} \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial y} + \frac{\partial \mathrm{v}_y}{\partial x}\right) \tag{3.3.22b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial x}\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \frac{\partial \mathrm{v}_y}{\partial y} \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial \mathrm{v}_y}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial y}\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \frac{\partial \mathrm{v}_z}{\partial z} \tag{3.3.22f}$$

Cylindrical coordinates

$$\tau_{rr} = 2\mu \frac{\partial \mathrm{v}_r}{\partial r} \tag{3.3.23a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\theta}{r}\right) + \frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta}\right) \tag{3.3.23b}$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial \mathrm{v}_r}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial r}\right) \tag{3.3.23c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta} + \frac{\mathrm{v}_r}{r}\right) \tag{3.3.23d}$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left(\frac{\partial \mathrm{v}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \mathrm{v}_z}{\partial \theta}\right) \tag{3.3.23e}$$

$$\tau_{zz} = 2\mu \frac{\partial \mathrm{v}_z}{\partial z} \tag{3.3.23f}$$

Spherical coordinates

$$\tau_{rr} = 2\mu \frac{\partial \mathrm{v}_r}{\partial r} \tag{3.3.24a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\theta}{r}\right) + \frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta}\right) \tag{3.3.24b}$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\phi}{r}\right) + \frac{1}{r \sin\theta}\frac{\partial \mathrm{v}_r}{\partial \phi}\right) \tag{3.3.24c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta} + \frac{\mathrm{v}_r}{r}\right) \tag{3.3.24d}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\mathrm{v}_\phi}{\sin\theta}\right) + \frac{1}{r \sin\theta}\frac{\partial \mathrm{v}_\theta}{\partial \phi}\right) \tag{3.3.24e}$$

$$\tau_{\phi\phi} = 2\mu\left(\frac{1}{r \sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial \phi} + \frac{\mathrm{v}_r}{r} + \frac{\mathrm{v}_\theta \cot\theta}{r}\right) \tag{3.3.24f}$$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x \qquad (3.3.26a)$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y \qquad (3.3.26b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad (3.3.26c)$$

Cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{{v_\theta}^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad (3.3.27a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad (3.3.27b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad (3.3.27c)$$