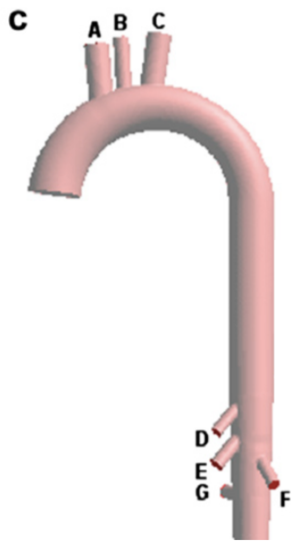
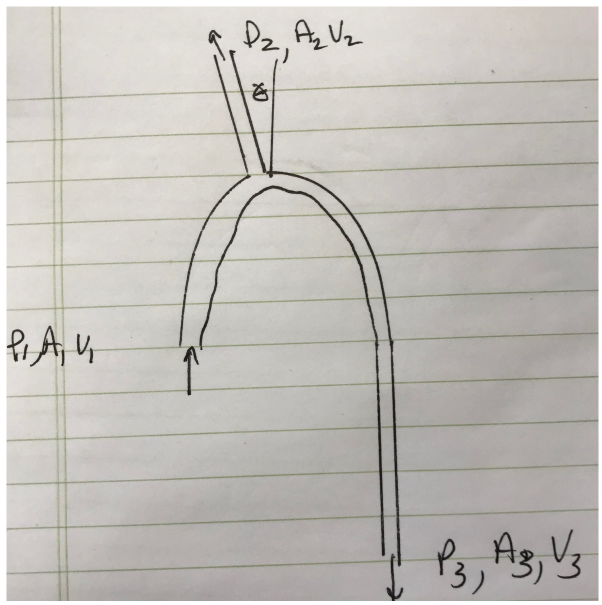
CBE 30357
Fall 2017
Test #2
11/9/17

1. Blood flow and forces on the human aorta. (30 points)

If you look for an image of the human aorta, you will find something like the figure on the right[1] (marked "c"). However, for this problem, let's start with the flow geometry on the right.



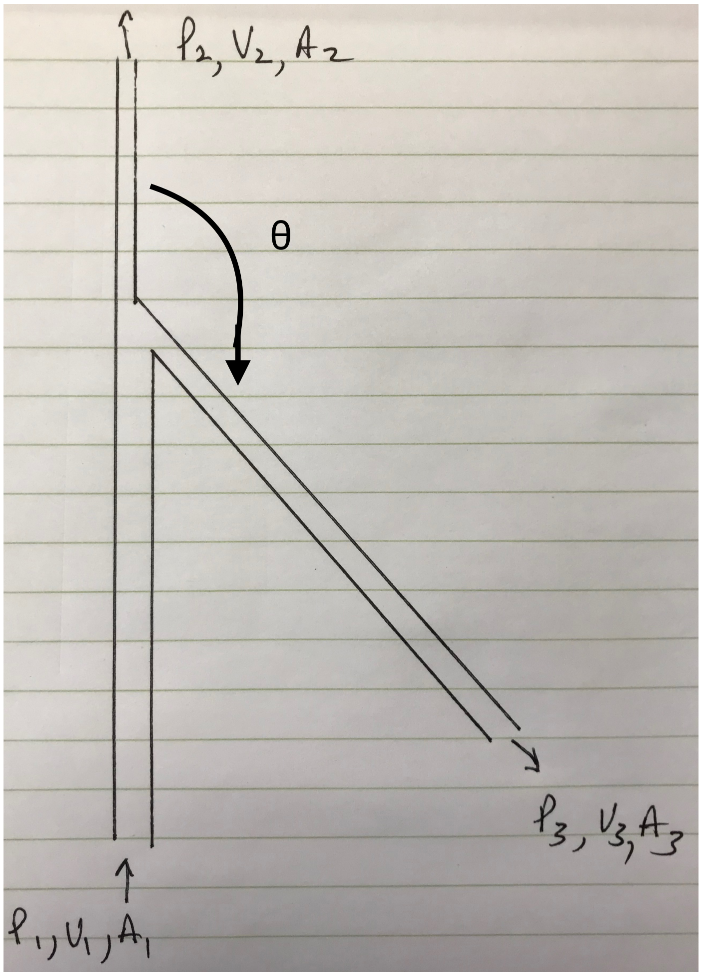


a. Give a relation for V_3 (the velocity in the descending aorta) in terms of V_1, V_2, and the areas.
b. Find the net force on this aorta in terms of the areas, velocities and pressures.
c. In a study of children and young adults Poutanen et al (2003) found that for the smallest children (with a body surface area of 0.5 to 0.75 m^2), the value of A_1 was 2.3 cm^2 and the value of A_3 was 0.49 cm^2. The average value of V_1 was 7.9 cm/s and the average value of V_3 was 8.3 cm/s. If $A_2 = A_1 - A_3$, what is the value of V_2?
d. If the total length of the aorta is 10 cm (5 cm for the curve and 5 cm for the straight tail), the viscosity of blood is 0.03 g/(cm-s) and the density of blood is 1.06 g/cm^3, give an <u>approximate</u> value of P_1-P_3.
e. Using your answer from *d*, explain the contributions of shear stresses to the total (net) force and the contributions due to other forces to the total net force of part *b*.
f. Is there any natural advantage to a specific value of θ, as compared to an arbitrary value?

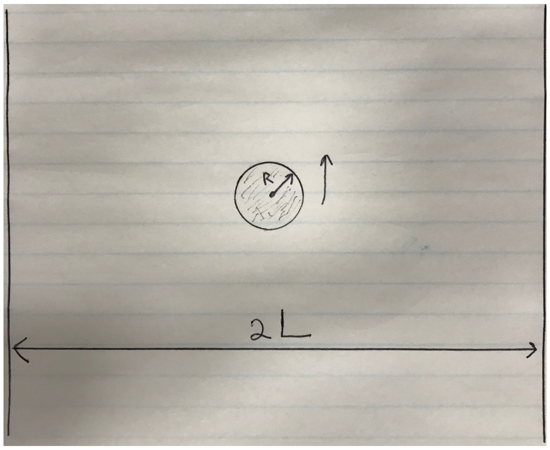
problem 1 continues….

[1] This image is from a computer modeling paper, but other cartoons are similar.

g. Now consider this slightly different flow configuration (that Nature did not choose!), what is the force necessary to hold this in place?



2. Rising spherical "drop" (or bubble) at low Reynolds number. (40 points)



We have examined the case of a solid sphere falling slowly in a otherwise quiescent liquid with viscosity, μ and density, ρ, that is of infinite extent. For this problem we consider a modification where the drop is now another fluid that has a density, ρ_p ($< \rho$) so that the drop will rise. The viscosity of the drop is μ_p.

The terminal (rise) velocity, V_s, is

$$V_p = \frac{1}{3}\frac{R^2 g}{\mu}(\rho_p - \rho)\,\frac{\mu + \mu_p}{\mu + \frac{3}{2}\mu_p} \quad ,$$

where g is the gravitation coefficient, 980 cm/s^2.

a. If a single drop of soybean oil, ρ_p = 0.917 g/cm^3 and μ_p = 0.5 g/cm-s with a radius of 0.1 mm is present in a vat of pure water, μ= 0.01 g/cm-s and ρ = 1 g/cm^3, what is the Reynolds number?

b. Find a simplification of this formula that would give a reasonably accurate answer if the soybean oil was replaced by a much more viscous oil, say "road tar".

c. Find a simplification of this formula that would give an accurate answer if the soybean oil was replaced by an air bubble?

d. What is the drag force on the drop or bubble that is contained in the equation above?

e. In the limit of Re->0, what components of the Navier-Stokes equations would be needed to solve for this flow field?

f. What boundary conditions are needed on the surface of the bubble or drop to solve for this flow field.

If we had solved for the flow field we would have found that outside the drop or bubble,

$$v_r\,(r,\,\theta) \;=\; U\,\mathrm{Cos}[\theta]\left(1 - \frac{\left(2 + 3\,\frac{\mu_s}{\mu}\right)}{2\,r\left(1 + \frac{\mu_s}{\mu}\right)} + \frac{\frac{\mu_s}{\mu}}{2\,r^3\left(1 + \frac{\mu_s}{\mu}\right)}\right) \quad .$$

e. If L = 3R, estimate the change in V_s compared to, say, L > 30 R.

f. Explain what is happening.

3. Several short answer problems. (30 points)

a. Viscosity of a suspension of "liquid" spheres in a second liquid.

The formula for the suspension viscosity, μ_s, of one liquid, μ_p dispersed in a second continuous liquid, μ is

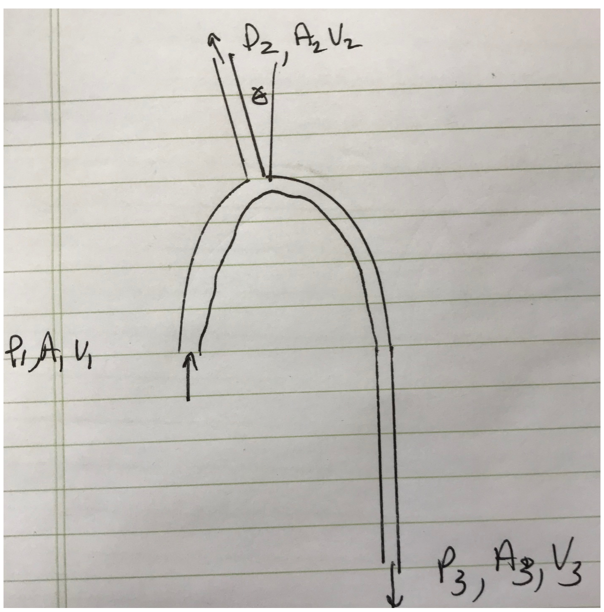
$$\frac{\mu_s}{\mu} = 1+\phi \, \frac{\mu + \frac{5}{2}\mu_p}{\mu + \mu_p}$$

, where ϕ is the volume fraction of particles.

Use this in a parallel plate (or any convenient) geometry to explain why Professor McCready recommended adding the flour to the (already mixed up) eggs and mixing, before adding a large quantity of milk when preparing the batter of a *German* pancake.

b. Heart catheterization in small children.

With reference to the diagram:



Suppose that it is necessary to simultaneously feed two catheter assemblies (i.e. long wires with probes on the ends) up the right branch of the (now *Abdominal* aorta) and around the "U" towards the heart (i.e., location 1).

If each catheter has a diameter, D_C, (which is less than 1/5 of D_3 = Sqrt[4A_3/π] or D_1 = Sqrt[4A_1/π]), estimate the increase in pressure drop, Δp/L in the right branch (D_3) as compared to the left branch (D_1) if the velocities are the same as in problem 1.

c. **Exhaling tobacco smoke.**

For various reasons, smoking is much less prevalent in movies today as are opportunities to even see someone smoking in public in the US. However, if you watch “old movies”, in particular “Film Noir” from the 1940’s, 50’s, etc. most of the main characters are constantly huffing and puffing.

An observation that can be made is that even when the actors are “serious” and inhale deeply, they still exhale a lot of smoke!

It is known that cigarette smoke has a average radius of 10^{-5} cm. The smallest lung passages have a radius of ~0.2 mm. A standard breath might take 2 sec to fill the lungs and 1.5 s to exhale.

<u>Briefly</u> explain why approximately 1/2 of the smoke comes back out and why the particle size of the exiting smoke is about the same as the entering smoke.

Some useful information could be: (note that η is the air viscosity the same as μ = *0.00018 g/cm-s*). The Boltzmann constant, *k*, in convenient units is 1.38 X 10^{-16} g $cm^2/(s^2$ K)

2 T.C. Carvalho et al. / International Journal of Pharmaceutics 406 (2011) 1–10

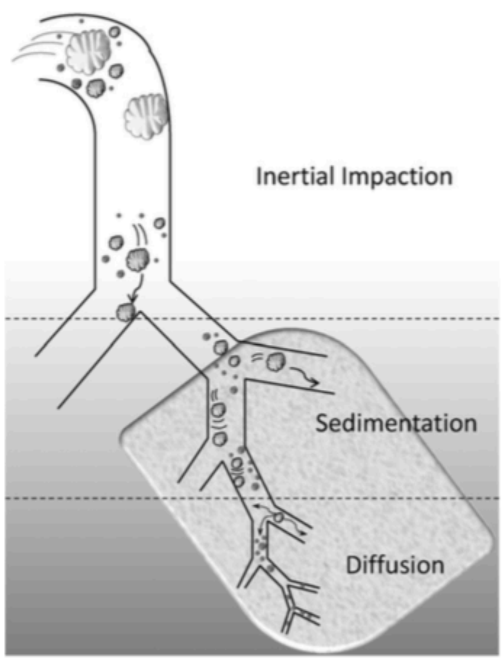


Fig. 1. Schematic diagram representing particle deposition in the lungs according to different mechanisms related to particle size: inertial impaction, sedimentation and diffusion. The diagram presents the smaller particles depositing in the lower airways as opposed to the larger airways. The GI tract is omitted in this diagram.

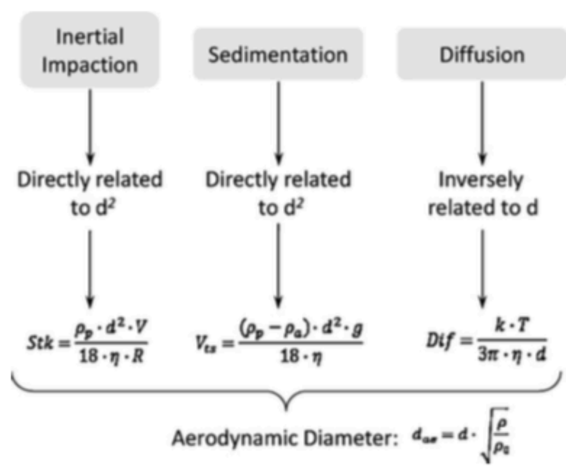


Fig. 2. The influence of particle size on deposition. *d*: particle diameter; *Stk*: Stokes number; ρ_p: particle density; *V*: air velocity; η: air viscosity; *R*: airway radius; V_{ts}: terminal settling velocity; ρ_a: air density; *g*: gravitational acceleration; *Dif*: diffusion coefficient; *k*: Boltzmann's constant; *T*: absolute temperature; d_{ae}: aerodynamic diameter; ρ_0: unity density.

B, mass, *m*, and velocity, *v*, according to Eq. (1) (Gonda, 2004):

$$S = B \cdot m \cdot v \quad (1)$$

The dimensionless Stokes’ number, *Stk*, more specifically describes the probability of particle deposition in the airways via impaction.

$$f \equiv \frac{\Delta p \; D}{2L \; \rho V^2} = \frac{\tau_w}{\frac{1}{2}\rho V^2},$$

$$Re \equiv \frac{DV\rho}{\mu}$$

$$f = \frac{16}{Re} \text{ (laminar flow)}$$

$$f = 0.079 \; Re^{-.25} \text{ (turbulent)}$$

$$Re_p \equiv \frac{D_p \; V_p \rho}{\mu} \text{ (particle Reynolds Number)}$$

$$C_D \equiv \frac{8}{\pi} \frac{F_D}{\rho V_p^2 D_p^2} \text{ (drag coefficient − drag force relation)}$$

Momentum equations for single flow inlet in + x direction.

$$-\rho \langle V_x V_x \rangle A_1 + \rho \langle V_2 V_2 \rangle A_2 \cos(\theta) = P_1 A_1 - P_2 A_2 \cos(\theta) + F_x$$

$$+\rho \langle V_2 V_2 \rangle A_2 \sin(\theta) = -P_2 A_2 \sin(\theta) + F_y$$

Surface Tension pressure jump across a curved interface:

$$\Delta p = \frac{2\gamma}{R}$$

$$\text{Area for circle} = \int_0^{2\pi} \int_0^R r \, dr \, d\theta$$

$$d_h = \frac{4 \; cross\text{−}section \; area}{perimeter}$$

$$D = \frac{k \; T}{6\pi\mu R}$$

where m is the total mass of the system. Combining Equations (4.3.5) and (4.3.7) yields the following relationship:

$$\frac{\partial \int_V \rho \mathbf{v} dV}{\partial t} + \int_S \mathbf{v}\rho(\mathbf{n} \cdot \mathbf{v}) dS = -\int_S p\mathbf{n} dS + \int_S \mathbf{n} \cdot \boldsymbol{\tau} dS + m\mathbf{g}. \qquad (4.3.8)$$

Equation (4.3.8) is the integral form of the equation of conservation of linear momentum. It is a vector equation and can be resolved into components in each of the three orthogonal axes of an appropriate coordinate system. Thus, there is one

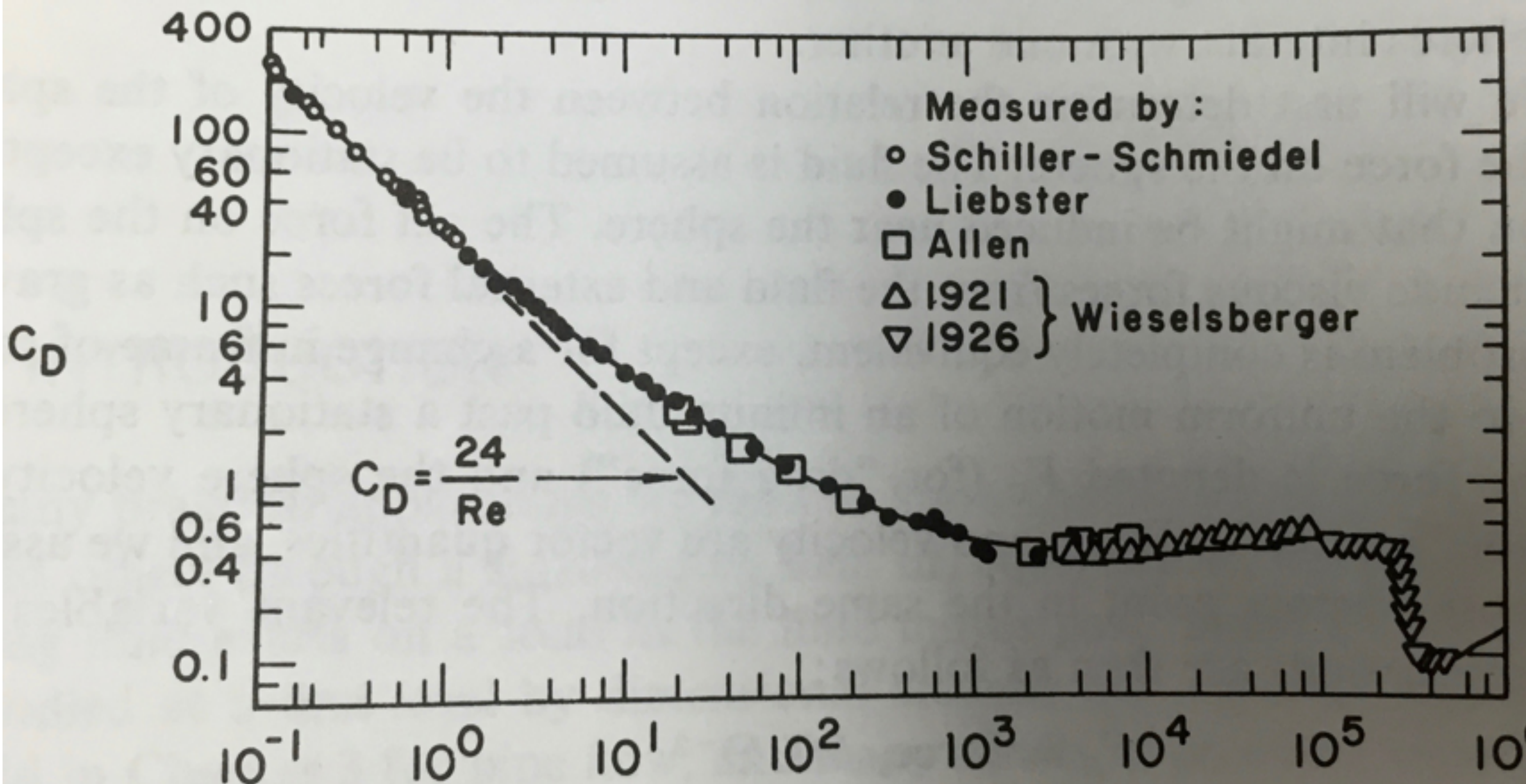


Figure 4-1. Drag coefficient as a function of Reynolds number for flow past a sphere. (Reproduced from H. Schlichting, *Boundary Layer Theory*, 6th ed., McGraw-Hill Book Company, New York, 1968, by

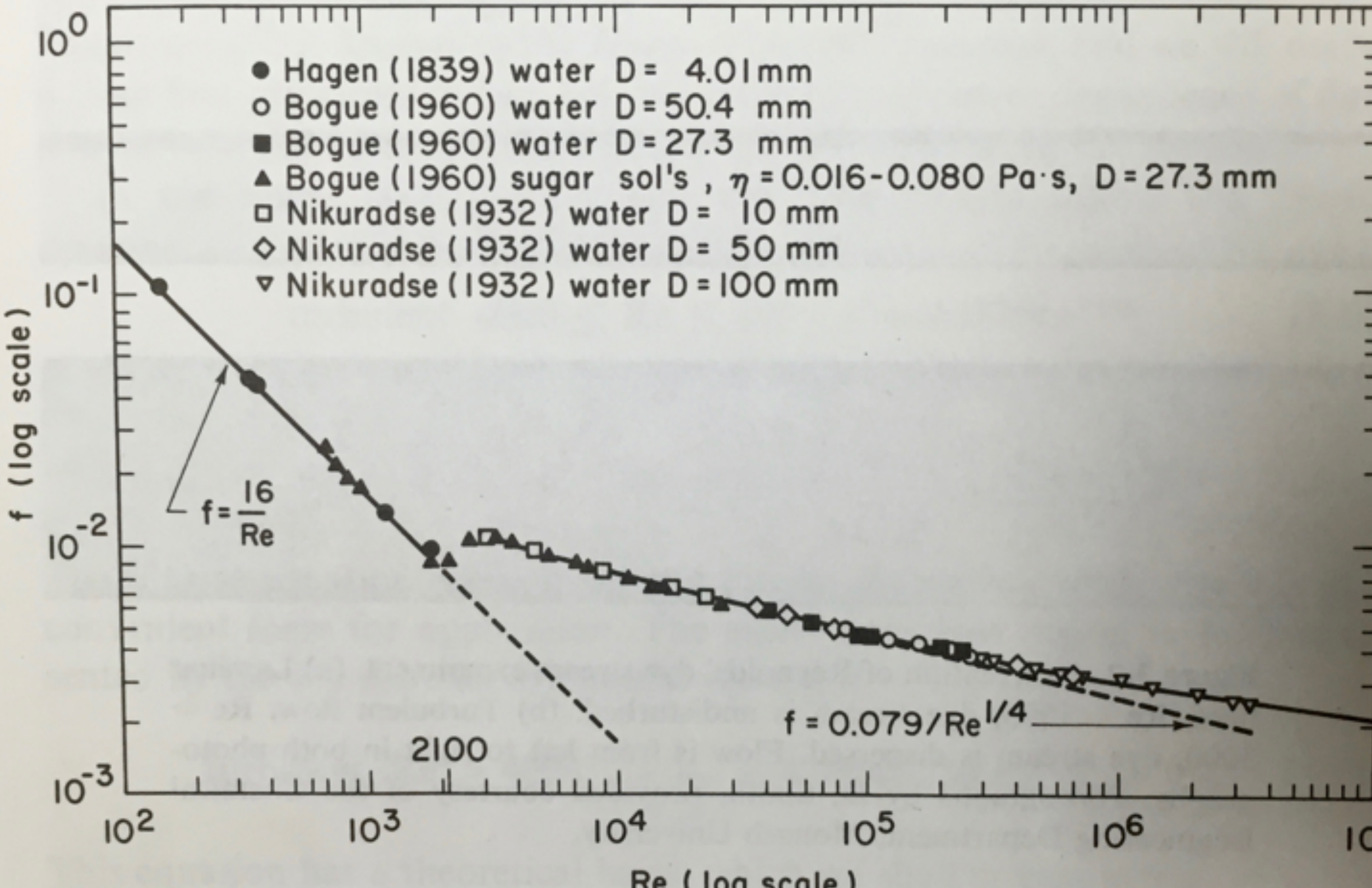


Figure 3-1. Friction factor as a function of Reynolds number for incompressible Newtonian fluids.

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}\right)$

TABLE 3.2

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \quad (3.3.18a)$$

y component

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \quad (3.3.18b)$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] \quad (3.3.18c)$$

Cylindrical coordinates

r component

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{zr}}{\partial z}\right] \quad (3.3.19a)$$

θ component

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] = \rho g_\theta - \frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right] \quad (3.3.19b)$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] \quad (3.3.19c)$$

Spherical coordinates

r component

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tau_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] \quad (3.3.20a)$$

TABLE 3.2

(*Continued*)
θ component
$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right] = \rho g_\theta - \frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}\right] \quad (3.3.20b)$$
ϕ component
$$\rho\left[\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right] = \rho g_\phi - \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\phi})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}\right] \quad (3.3.20c)$$

TABLE 3.3

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \frac{\partial \mathrm{v}_x}{\partial x} \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial y} + \frac{\partial \mathrm{v}_y}{\partial x}\right) \tag{3.3.22b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial x}\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \frac{\partial \mathrm{v}_y}{\partial y} \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial \mathrm{v}_y}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial y}\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \frac{\partial \mathrm{v}_z}{\partial z} \tag{3.3.22f}$$

Cylindrical coordinates

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \tag{3.3.23a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{3.3.23b}$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \tag{3.3.23c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \tag{3.3.23d}$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \tag{3.3.23e}$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \tag{3.3.23f}$$

Spherical coordinates

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \tag{3.3.24a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{3.3.24b}$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r \sin\theta}\frac{\partial v_r}{\partial \phi}\right) \tag{3.3.24c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_r}{r}\right) \tag{3.3.24d}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \tag{3.3.24e}$$

$$\tau_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) \tag{3.3.24f}$$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x \quad (3.3.26a)$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y \quad (3.3.26b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad (3.3.26c)$$

Cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{{v_\theta}^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \quad (3.3.27a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \quad (3.3.27b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad (3.3.27c)$$

TABLE 3.4

(Continued)
Spherical coordinates

r direction

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = -\frac{\partial p}{\partial r} + \rho g_r$$
$$+ \mu\left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2} - 2\frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta - \frac{2}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right] \quad (3.3.28a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta v_r}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta \quad (3.3.28b)$$

ϕ direction

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\phi \quad (3.3.28c)$$

Surface Tension pressure jump across a curved interface:

$$\Delta p = \frac{2\gamma}{R}$$

$$\text{Area for circle} = \int_0^{2\pi}\int_0^{R} r\,dr\,d\theta$$