CBE 30357
Fall 2017
Final Exam
December 13, 2017

1. Coagulation caused by a restriction in a blood vessel? (65 points)

In a recent publication, Rukhlenko et al. (2015) claim that increased wall shear stress at the walls of an artery, such that can occur near a stenosis (i.e., a local reduction in cross section area, probably from the build up of plaque) could change the permeability of the vessel wall to "procoagulants" which in the extreme can lead to coagulation within the blood vessel. This mechanism, where such clotting factors would normally be permeating out of the vessel — but do not —, has been observed at shear rates as low as 1000/s. This value contrasts a more widely recognized mechanism for clot formation, "platelet aggregation, which does not occur until much higher shear rates, ~5400/s.

We would like to examine this claim of reduced permeability leading to clotting and see if we can conclude anything based on our current understanding of fluid mechanics.

Let's start with what we know well.

Consider blood flow as fully-developed laminar flow of a Newtonian fluid in a circular tube of radius, *R*. It has a volumetric flow, *Q*.

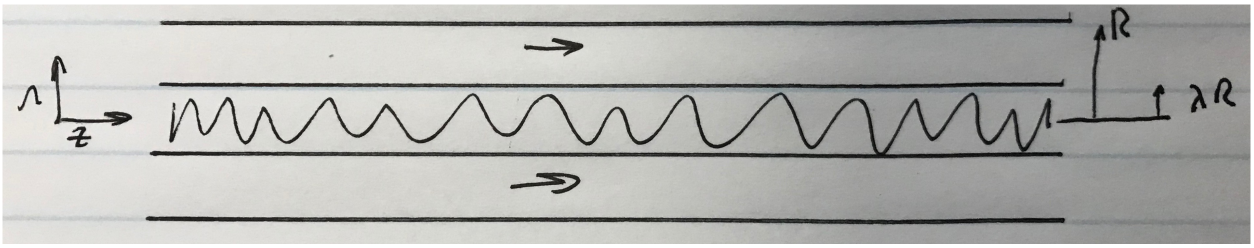
a. Write down the relevant terms of the Navier-Stokes equations that will be solved to describe a fully-developed flow.
b. Write down the boundary conditions needed to solve for the flow field.
c. Solve for the velocity field and sketch the profile.
d. Find an expression that relates the pressure gradient and the volumetric flow rate.
e. Find an expression for the shear stress everywhere in the blood vessel.

Now let's examine a real coronary artery. The vessel has a diameter of 0.4 cm. The average velocity of the blood in the artery when the phase of the heart is at its maximum is 20 cm/s. Use a viscosity value of μ = 0.035 g/(cm-s) for blood.

f. What is the maximum value of the shear stress and where does it occur?
g. The shear rate is $\partial v_z/\partial r$. What the maximum value of the shear rate?
h. For these flow conditions, what value of radius would give a shear rate of 5000/s? (i.e., within the range of causing platelet aggregation?
i. For these conditions, what value of radius would cause a shear rate of 1000/s, (alleged to be associated with loss of permeability?)
j. The time-averaged velocity of blood is only 7 cm/s. Does this change your assessment of the likelihood that <u>lowered vessel permeability</u> is a significant clotting risk?

2. **Examination of coronary artery by catheterization. (35 points)**

Suppose that it is determined that it is necessary to examine various points around the heart by catheterization. We would like to quantify the increased stress and shear rate associated with this procedure.



The radius of the artery is *R* and the radius of the catheter (a solid wire for all intents) is λR. So blood flow is now confined to the annular region between λR and R.

a. Revisit problem 1a, and write the boundary conditions and non-zero Navier-Stokes terms that are necessary to solve the flow in the annular region around the catheter? Note that the catheter could be moving at velocity V_c that could be positive or negative.

The solution for the velocity profile that you would seek is:

Out[48]=
$$\frac{4\,\mu\,V_c\,\log\left(\frac{r}{R}\right) + \text{dpdz}\,r^2\,\log(\lambda) + \text{dpdz}\,R^2\left(\lambda^2\,(-\log(r)) + \log(r) + \lambda^2\,\log(R) - \log(\lambda\,R)\right)}{4\,\mu\,\log(\lambda)}$$

Note that "log" means natural log.

You realize that with enough time, you could continue and get an average velocity, a shear stress profile and find out how much the catheter restricts the flow. You don't have this much time!

c. Find a simplifying approximation to get the <u>increased pressure drop</u> as a function of the diameter of catheter for the case where $V_c = 0$.

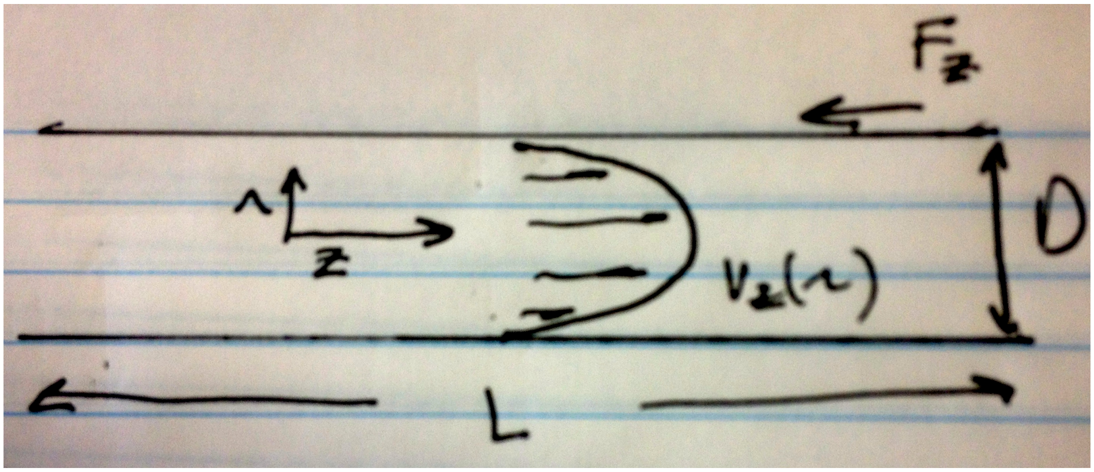
d. Under what conditions is your solution valid?

Use your new solution for the final part….

e. Suppose we are examining the same artery as in problem 1 with an average velocity of 20 cm/s. If the catheter has a diameter of 1mm, by what factor does the pressure gradient have to increase to keep the blood flow the same as would be occurring in the absence of the catheter?

3. **Forces on pipes and fittings (40 points)**

a. For <u>"laminar" flow of Newtonian fluid</u> in a circular pipe, find an algebraic expression for F_z, the force on the outside of the pipe necessary to keep the tube in place.



b. Explain how the force varies with Reynolds number

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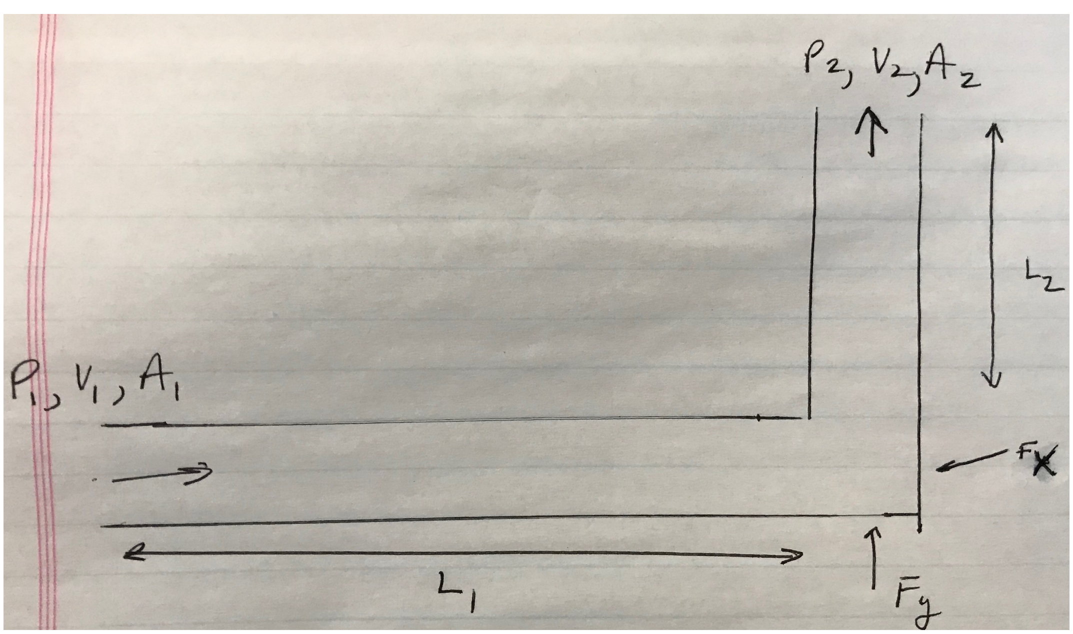
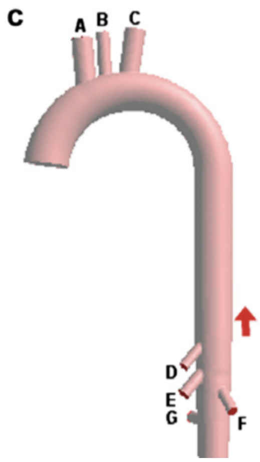
c. Water is flowing through the pipe and fitting shown in the figure below. Find a good estimate for F_x and F_y if:

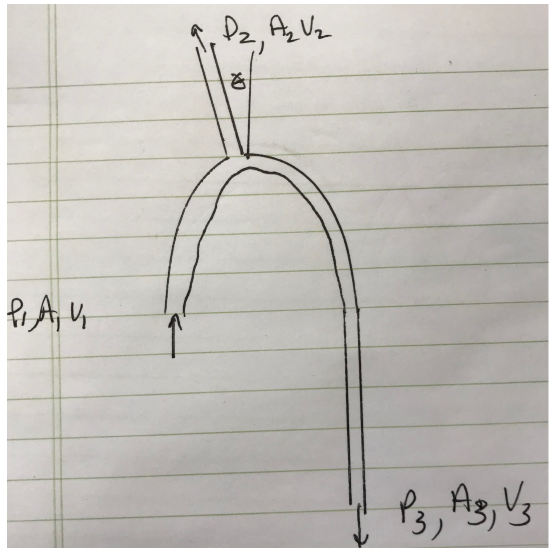
P_1 = 200,000 N/m^2, = 200,000 kg /(m-s)
V_1= 3 m/s
A_1= 0.0000785 m^2 (diameter = 1 cm)
A_2 = 0.0000636 m^2 (diameter = 0.9 cm)
L_1 = 2 m,
L_2 ~=0
μ=0.001 kg/(m-s)
ρ=1000 kg/m^3



4. **Entrance effects in arterial flow. (40 points)**

Recall the “aorta” from test #2.



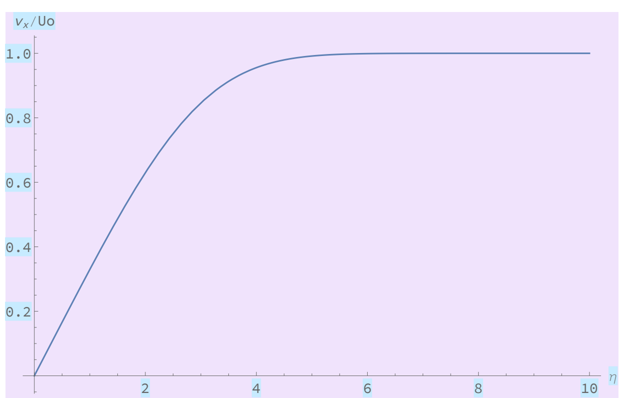


We would like to know if for this very long artery, ~25 cm, if entrance effects contribute significantly to the pressure drop.

We will consider older children for this calculation. Use the approximation that the aorta has a constant radius of 1.5 cm and blood velocity of 20 cm/s.

For this (more or less) steady laminar flow, it has been suggested that boundary-layer results could provide some insight that could answer this question.

$$\eta \equiv \frac{y}{\sqrt{\frac{x\nu}{U}}}$$



a. Find an estimate for the distance necessary for the flow to become fully-developed.
b. Compare the value of the wall shear stress at the entrance to what a steady state laminar flow would achieve sufficiently far downstream. Is the shear stress in the entrance region higher?
c. Does the answer to part b confirm the the pressure drop will be higher for developing flows compared to full-developed flows?

5. **Production of pharmaceuticals by lyophilization (a.k.a. freeze-drying) (30 points)**

While “freeze drying” was formerly a “fad” in the industry to make, (before Keurig), e.g., instant coffee, it continues to be critical to production of many antibiotics and other delicate pharmaceutical compounds.

After the synthesis and separation steps are completed, it may be necessary to remove water, but leave behind a “powder” that can be readily re-wetted, (perhaps with a sterile saline + buffer if the drug is to be administered intravenously). The pharmaceutical compound is sensitive to heat so that a normal drying process cannot be performed.

Instead, the water-drug mixture is frozen and the ice is sublimed leaving a dry powder with a useful “shelf-life”. You can probably figure out that this is not by doing the synthesis in a very northern climate, putting the suspension in a bucket outside and waiting for the ice to evaporate!

Normally the suspension is sprayed, through a nozzle to make drops of a prescribed size, into a chamber that is at a pressure significantly below 1 ATM and (to the extent that T is well-defined) is sufficiently cold to effect rapid ice formation.

Freezing and sublimation is completed while the drops are falling onto a collection surface. It is this last step that you need to design.

Suppose that the rate of water removal is such that for drops/particles in the size range that will be ideal, takes 5 s. You need to prescribe a drop size that makes particles that will fall through the 1 m distance in at least a little longer than 5 s.

You can assume that gravity acceleration is 980 cm/s^2. The density of the pharmaceutical and the density of water are both 1 g/cm^3. The density of the gas in the freezing chamber is only 0.1 ATM or about 0.00013 g/cm^3 (@ -40C). Interestingly, the pressure does not significantly affect the gas viscosity, but at this temperature you can use a value of 0.00015 g/(cm-s).

A. If the amount of water removed is such that the drop diameter changes only by a few percent as it dries, suggest a drop diameter that will work for this process.

$$f \equiv \frac{\Delta p\ D}{2L\ \rho V^2} = \frac{\tau_w}{\frac{1}{2}\rho V^2},$$

$$Re \equiv \frac{DV\rho}{\mu}$$

$$f = \frac{16}{Re}\ \text{(laminar flow } pipe\text{)}$$

$$f = 0.079\ Re^{-.25}\ \text{(turbulent, pipe)}$$

$$Re \equiv \frac{hV\rho}{\mu}\ (rectangular\ channel)$$

$$f = \frac{6}{Re}\ (rectangular\ channel)$$

$$Re_p \equiv \frac{D_p\ V_p\rho}{\mu}\ \text{(particle Reynolds Number)}$$

$$C_D \equiv \frac{8}{\pi}\frac{F_D}{\rho V_p^2 D_p^2}\ \text{(drag coefficient} - \text{drag force relation)}$$

Momentum equations for single flow inlet in + x direction.

$$-\rho\langle V_x V_x\rangle A_1 + \rho\langle V_2 V_2\rangle A_2\cos(\theta) = P_1A_1 - P_2A_2\cos(\theta) + F_x$$

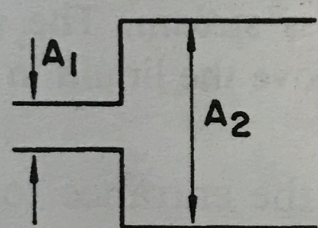
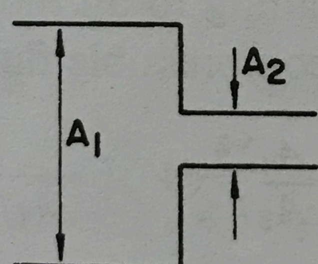
$$+\rho\langle V_2 V_2\rangle A_2\sin(\theta) = -P_2A_2\sin(\theta) + F_y$$

$$\left(\frac{V_2^2}{2} + gh_2 + \frac{P_2}{\rho}\right) - \left(\frac{V_1^2}{2} + gh_1 + \frac{P_1}{\rho}\right) = \delta\ W_s - l_v$$

$$l_v = \sum \frac{1}{2} K\ V_2^2\ \ or\ \ \frac{2\ L\ f\ V^2}{D}$$

TABLE 5-1

LOSSES IN FITTINGS AND VALVES FOR TURBULENT FLOW[a]

Fitting or valve		*Velocity heads lost, K_f*
90° elbow, standard		0.75
90° elbow, square		1.3
Coupling		0.04
Gate valve		
Open		0.17
Half-open		4.5
Globe valve, bevel seat		
Open		6.4
Half-open		9.5
Sudden expansion		$\left(\frac{A_2}{A_1} - 1\right)^2$
Sudden contraction		$\left(\frac{2}{m} - \frac{A_2}{A_1} - 1\right)^2$ m is the root of the quadratic $\frac{1 - m(A_2/A_1)}{1 - (A_2/A_1)^2} = \left(\frac{m}{1.2}\right)^2$
Rounded entrance		0.05

[a]The result for the sudden expansion is derived in Sec. 6.2. The result for the sudden contraction is from Martin, *Chem. Eng. Educ.*, Summer 1974, p. 138. Other values are from *Perry's Handbook*.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0,$$

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = \rho U(x) \frac{dU(x)}{dx} + \mu \frac{\partial^2 v_x}{\partial y^2}.$$

$$\frac{\rho}{2} dv^2 + dp + \rho g dz = 0.$$

1b) yields

$$\frac{1}{2}\rho v^2 + p + \rho g z = \text{constant}.$$

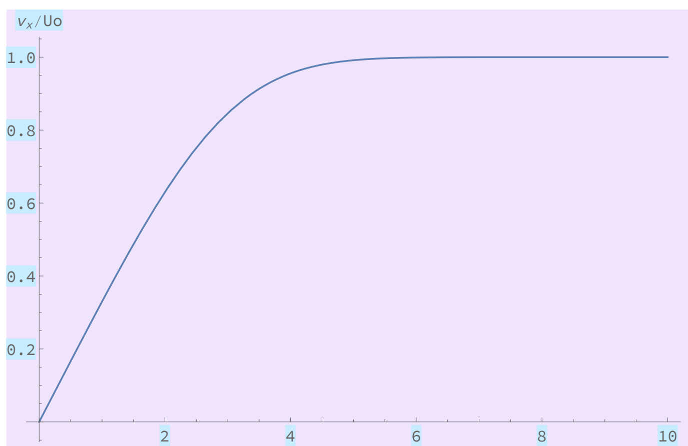
$$P(x) + \frac{1}{2}\rho U(x)^2 = \text{constant}.$$

to be known. Taking the deriv
following expression for the p

$$\frac{dP}{dx} = -\rho U(x) \frac{dU(x)}{dx}.$$

$$\eta \equiv \frac{y}{\sqrt{\frac{x\nu}{U}}}$$

$$\frac{1}{2} f[\eta]\, f''[\eta] + f^{(3)}[\eta]$$



Surface Tension pressure jump across a curved interface:

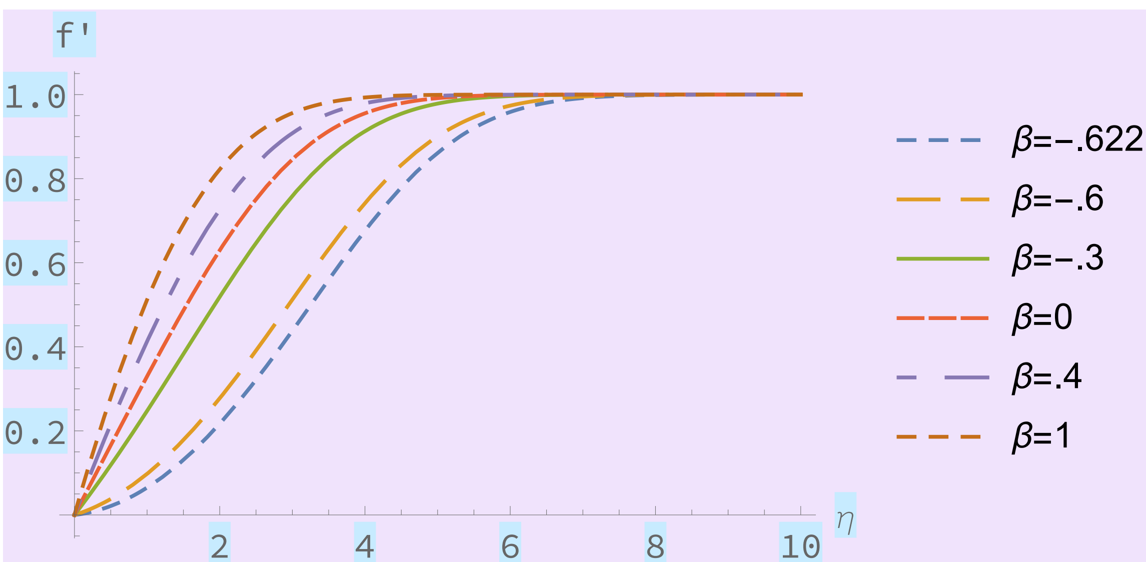
$$\Delta p = \frac{2\gamma}{R}$$

$$\text{Area for circle} = \int_0^{2\pi}\int_0^R r\,dr\,d\theta$$

$$d_h = \frac{4\ cross\text{−}section\ area}{perimeter}$$

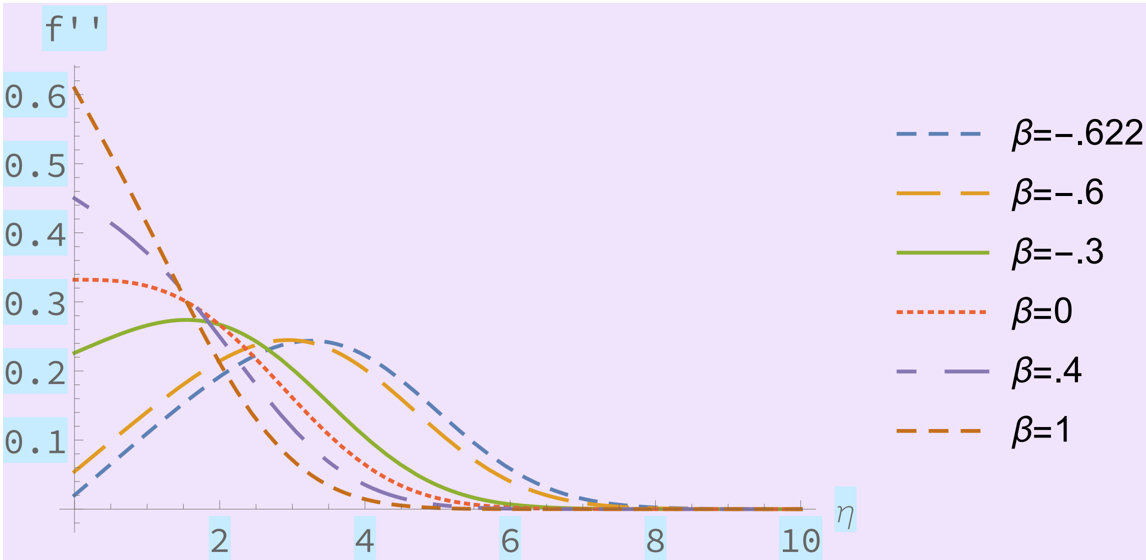
$$D = \frac{k\,T}{6\pi\mu R}$$

$k = 5.6704\times10^{-5}$ erg cm^{-2} s^{-1} K^{-4} = 1.38×10^{-16} g cm^2/(s^2 K)



β=0 if plate is parallel to flow

$$v_x = U_0 f'(\eta)$$



$$\tau = \mu \frac{\partial v_x}{\partial y} = \mu U_0 \left(\frac{U_0 \rho}{x\mu}\right)^{1/2} f''(0)$$

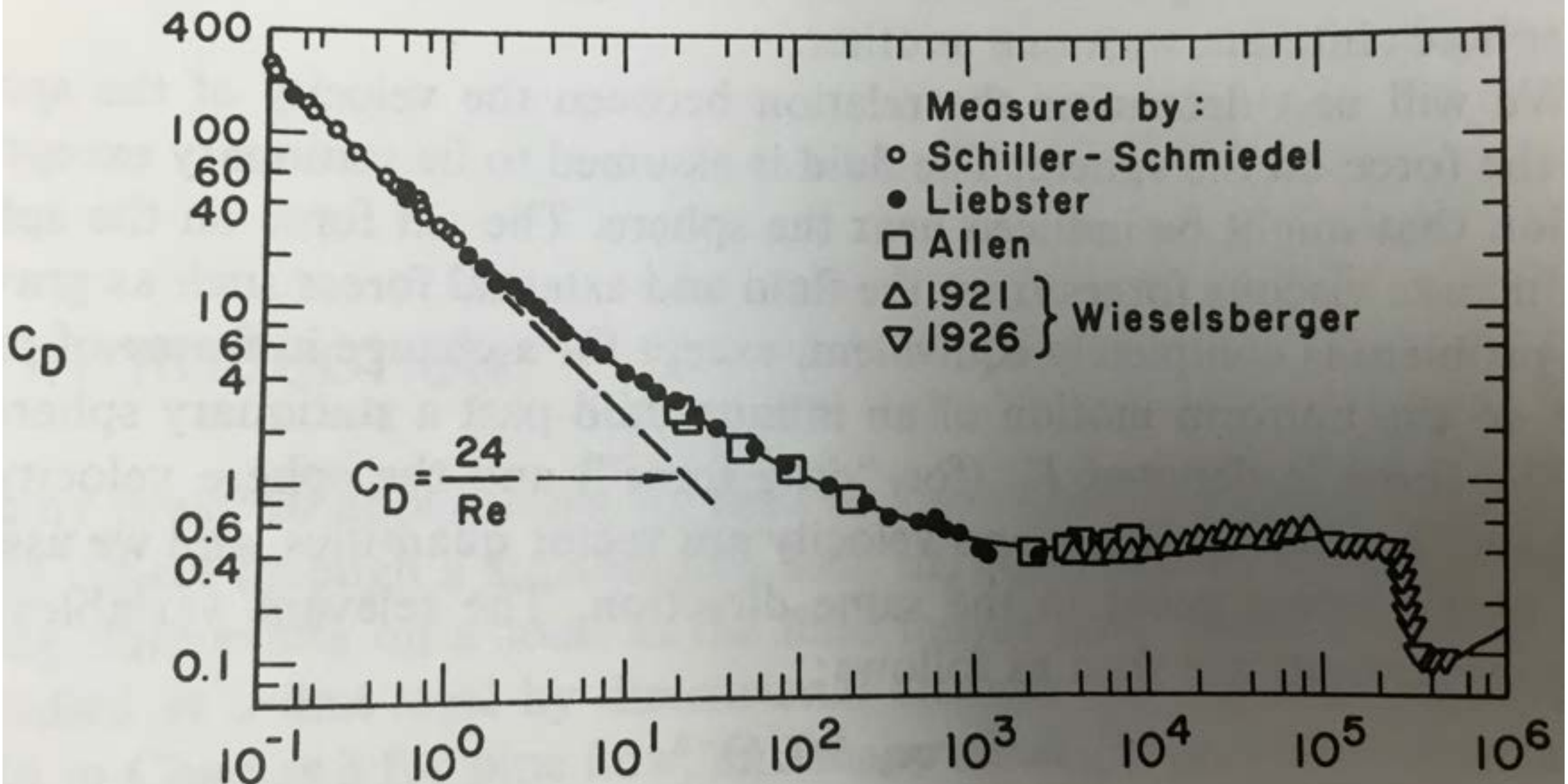


Figure 4-1. Drag coefficient as a function of Reynolds number for flow past a sphere. (Reproduced from H. Schlichting, *Boundary Layer Theory*, 6th ed., McGraw-Hill Book Company, New York, 1968, by

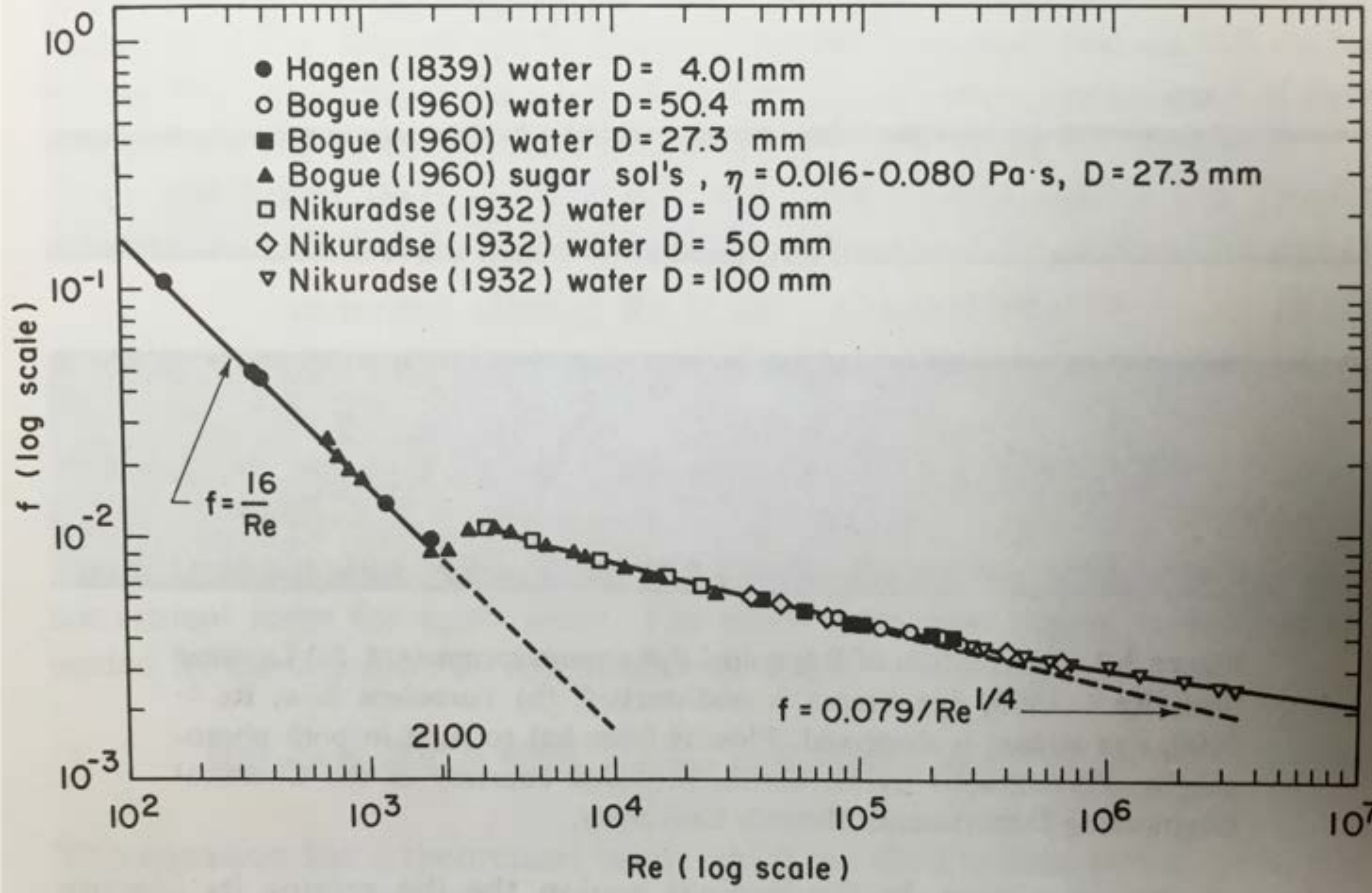


Figure 3-1. Friction factor as a function of Reynolds number for incompressible Newtonian fluids.

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}\right)$

TABLE 3.2

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \quad (3.3.18a)$$

y component

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \quad (3.3.18b)$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] \quad (3.3.18c)$$

Cylindrical coordinates

r component

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{zr}}{\partial z}\right] \quad (3.3.19a)$$

θ component

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] = \rho g_\theta - \frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right] \quad (3.3.19b)$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] \quad (3.3.19c)$$

Spherical coordinates

r component

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tau_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] \quad (3.3.20a)$$

TABLE 3.2

(*Continued*)

θ component

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right] = \rho g_\theta - \frac{\partial p}{\partial \theta}$$

$$+\left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\tau_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}\right] \quad (3.3.20b)$$

ϕ component

$$\rho\left[\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right] = \rho g_\phi - \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$

$$+\left[\frac{1}{r^2}\frac{\partial(r^2\tau_{r\phi})}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}\right] \quad (3.3.20c)$$

TABLE 3.3

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \frac{\partial \mathrm{v}_x}{\partial x} \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial y} + \frac{\partial \mathrm{v}_y}{\partial x}\right) \tag{3.3.22b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial \mathrm{v}_x}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial x}\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \frac{\partial \mathrm{v}_y}{\partial y} \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial \mathrm{v}_y}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial y}\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \frac{\partial \mathrm{v}_z}{\partial z} \tag{3.3.22f}$$

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial\rho}{\partial t} = -\left(\frac{\partial\rho v_x}{\partial x} + \frac{\partial\rho v_y}{\partial y} + \frac{\partial\rho v_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial\rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial\rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial\phi}\right)$

Cylindrical coordinates

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \tag{3.3.23a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{3.3.23b}$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \tag{3.3.23c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_r}{r}\right) \tag{3.3.23d}$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \tag{3.3.23e}$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \tag{3.3.23f}$$

Spherical coordinates

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \tag{3.3.24a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{3.3.24b}$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left(r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\right) \tag{3.3.24c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_r}{r}\right) \tag{3.3.24d}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \tag{3.3.24e}$$

$$\tau_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) \tag{3.3.24f}$$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x \quad (3.3.26a)$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y \quad (3.3.26b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad (3.3.26c)$$

Cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \quad (3.3.27a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \quad (3.3.27b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad (3.3.27c)$$

Table 3.4 (continued)

TABLE 3.4

(*Continued*)

Spherical coordinates

r direction

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = -\frac{\partial p}{\partial r} + \rho g_r$$
$$+ \mu\left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2} - 2\frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta - \frac{2}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right] \quad (3.3.28a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta v_r}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta \quad (3.3.28b)$$

ϕ direction

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\phi \quad (3.3.28c)$$