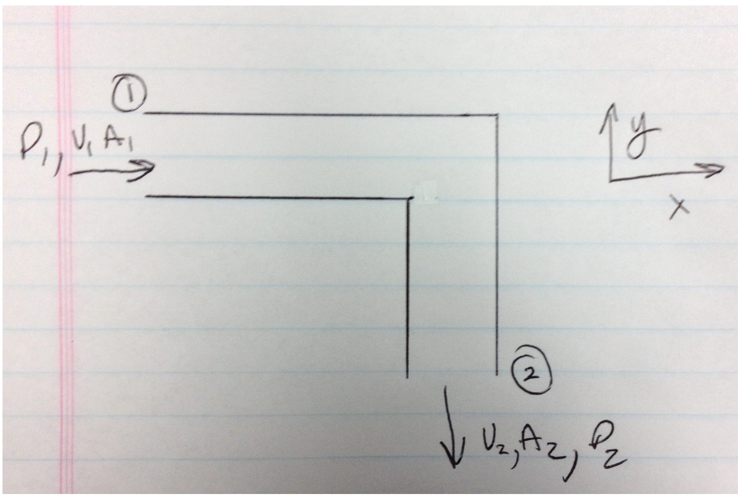
CBE 30357
Fall 2013
Test#2
November 21, 2013

1. Flow into a "elbow" (40 points)

Consider the flow of a liquid into the elbow shown below. In this problem the (constant) density is ρ, the average inlet velocity is v_1 and the inlet area is A_1.



Consider first that area does not change around the bend and that the flow does not have any viscous losses.

a. Explain if there is any pressure change from point 1 to 2?
b. Find the force <u>on the bend</u> caused by the fluid.

Now allow for the possibility of viscous losses

c. If $l_v = K v^2$ and K = 0.3 for this bend, find a relation for the pressure difference from location 1 to location 2.
d. For this case, calculate the force on the bend. Explain any changes from b.

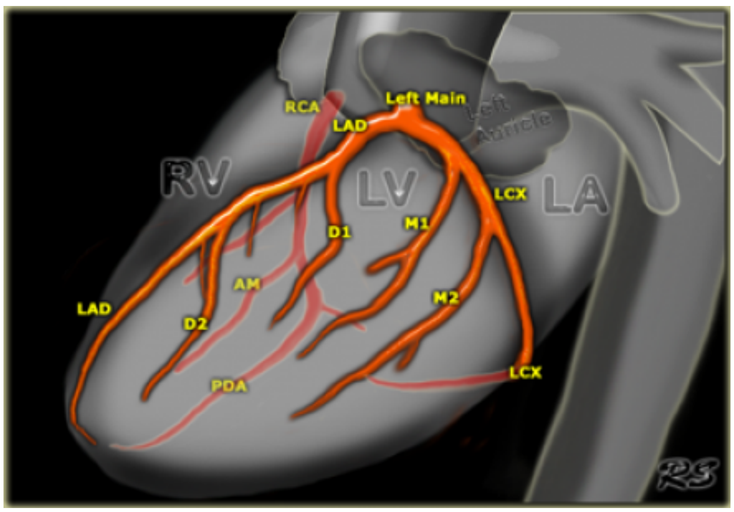
Now consider the case for which the exit area A_2, at location 2 is <u>smaller</u> than the inlet area, A_1.

d. For the case where K = 0.3 (A_1/A_2) v_2^2, find a relation for the pressure difference P_1-P_2.
e. Yet again… find the force of the fluid on the bend and explain what happened, in particular look at the y direction force.

2. Blood Flow around the heart. (40 points)

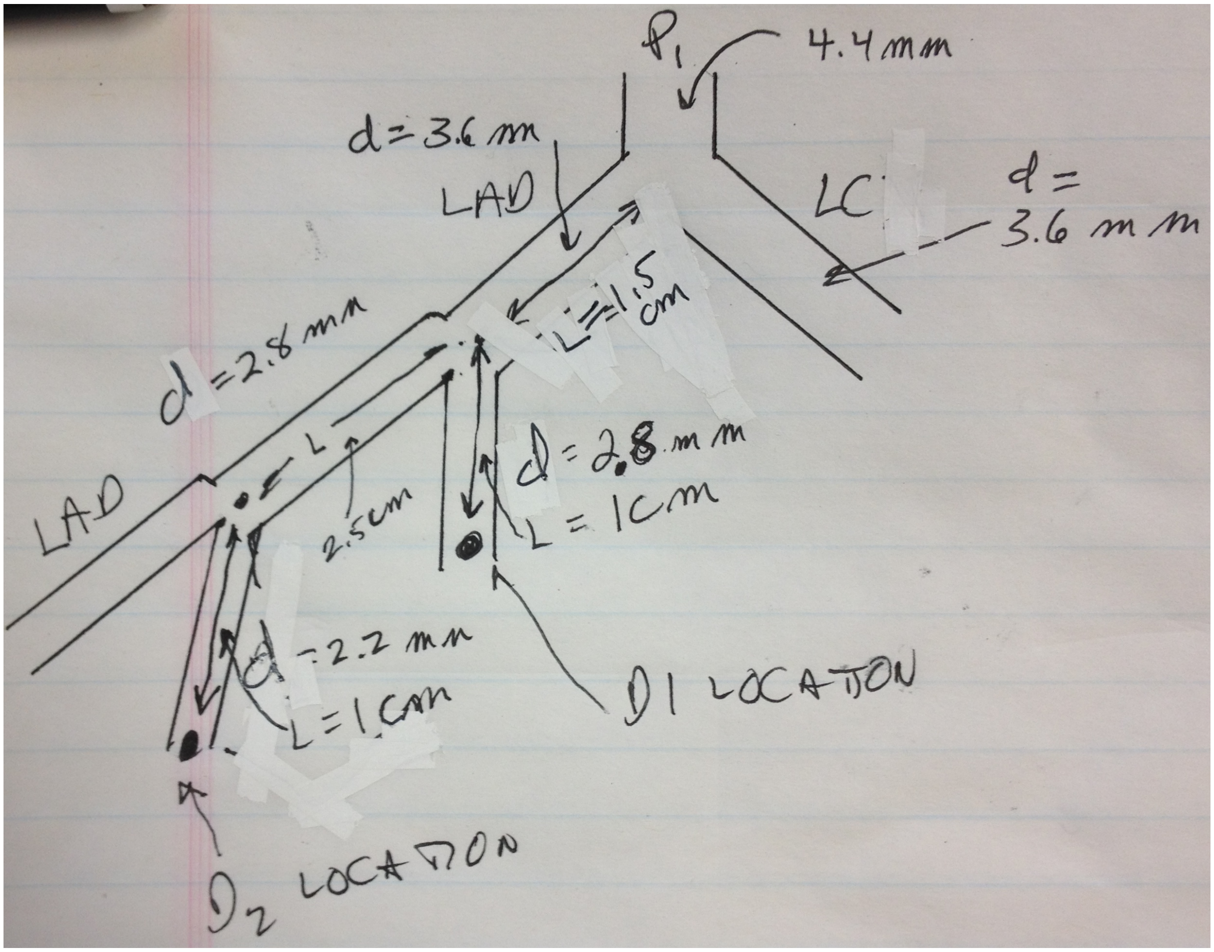
Blood flow to the heart muscle itself is described as having “limited redundancy”, hence the danger that blockage of a single artery will lead to significant damage.

Consider flow in the left branch of the heart. The left main artery, which is directly connected to the aorta, branches into the left anterior descending (LAD) and the circumflex (LCX) arteries. The LAD and LCX arteries have about the same diameter, 3.6 mm, and the left main is about 4.4 mm. (So the branching misses Murray’s law by a bit.) You can probably imagine that given that the arteries are this small, a small blockage or overall narrowing can be quite a problem for the person whose heart has these conditions!



Because the left main connects directly to the aorta, the pressure that could be available for flow is well known. There must be some ability for the LM, LAD and LCX to dilate or contract (change diameter) to adjust blood flow to the heart that could be different from the overall output of the heart (although there would be a relation between pumping output and cardiac artery flow needs.)

We are interested in the severity of flow consequences and heart damage that could occur if some blockage occurred.



a. Since the LM comes from the aorta, we know the pressure at the beginning of the LM. It is short so that this pressure is about the same just before the branch. Let's pick 150 mmHg (=200,000 dyne/cm^2) for this pressure. The losses for a branching geometry are 0.2 v^2, and the LM velocity is about 8 cm/s. Calculate the pressures at "D1" and "D2" which are 1 cm into the D1 and D2 arteries.

b. If the person has some coronary artery disease, the diameter for LAD could be reduced to 2.5 mm, D1 and the LAD branch would now be 1.8mm and the D2 branch would be 1.1 mm. Now calculate the pressures at locations D1 and D2. Can the flow be maintained?

You have probably heard "stories" about people going in for checkups and not being allowed to leave without either bypass surgery or catheterizations to insert stents into the the cardiac arteries.

Suppose that in the middle of the LAD there is a blockage such that only 0.04 fraction of the area is available for flow. For this irregular flow shape the losses are $K = 10\left(\frac{A_1}{A_2}\right)^2 v^2$. For this part of the problem consider that A_1 is for a clear artery with a diameter of 3.6 mm. A_2 = .04 A_1, v is for the clear part of the artery.

c. Now what are the pressures at D1 and D2?
d. Can the same flow be maintained?

Blood has a viscosity, μ, of 3.5 cP (0.035g/(cm-s)), and a density, ρ, of 1.05 g/cm^3. The gravity constant, g, is 980 cm/s^2. Here are some useful numbers!

artery	diameter (cm)	length (cm)	flowrate (cc/s)	velocity (cm/s)	Re	f	lv(cm^2/ s^2)
LM	0.44	0	1.22	8.00	105.60	0.15	0.0
LAD (before)	0.36	1.5	0.6	5.90	63.69	0.25	1709.4
D1	0.28	1	0.3	4.87	40.95	0.39	827.3
LAD (after)	0.28	2.5	0.3	4.87	40.95	0.39	2068.3
D2	0.22	1	0.15	3.95	26.06	0.61	559.4
LM	0.44	0	1.22	8.00	105.60	0.15	0.0
LAD (before)	0.25	1.5	0.6	12.23	91.72	0.17	21947.5
D1	0.18	1	0.3	11.80	63.69	0.25	18233.8
LAD (after)	0.18	2.5	0.3	11.80	63.69	0.25	45584.5
D2	0.1	1	0.15	19.11	57.32	0.28	139538.7

For this problem, we will consider blood flow to be steady laminar flow such that,

$Q = \dfrac{\Delta p \pi R^4}{8 \mu L}$, where Q is the volumetric flow rate, R is the tube radius, Δp is the pressure drop for length L, and μ is the fluid viscosity.

3. Scaling of flow in tubes and bends (20 points)

Consider flow in a straight, circular tube for which the Reynolds number is 1000, so that the flow is laminar. The liquid is water with a viscosity, $\mu = 0.001$ kg/(m-s) and density, $\rho = 1000$ kg/m^3. The tube length is 1m and the diameter, D is 0.01m.

a. What is the average velocity value?
b. What is the pressure drop in this tube?
c. Determine the scaling relation between and V, the average velocity, that is, to what power of V does Δp vary?

Now consider that this tube is connected to an elbow which has viscous losses of $l_v = K v^2 \quad K = 0.3$

e. What is the pressure change across the elbow for this flow?
f. Determine the scaling relation between Δp and V, the average velocity, that is, to what power of V does Δp vary?
g. Consider your answers to c and f, are they consistent? Explain.
h. Would the answer to g change if the Reynolds number were 0.1? Explain.

Formulae of interest

$$\rho < v_2^2 > \boldsymbol{A}_2 - \rho < v_1^2 > \boldsymbol{A}_1 + P\boldsymbol{A}_2 - P\boldsymbol{A}_1 = -\boldsymbol{F}$$

$$\rho < v_2 > A_2\boldsymbol{v}_2 - \rho < v_1 > A_1\boldsymbol{v}_1 + P\boldsymbol{A}_2 - P\boldsymbol{A}_1 = -\boldsymbol{F}$$

$$\left(\frac{< v_2 >^2}{2} + gh_2 + \frac{P_2}{\rho}\right) - \left(\frac{< v_1 >^2}{2} + gh_1 + \frac{P_1}{\rho}\right) = \delta W_s - l_v$$

$$\sum_i \rho \langle v_i \rangle A_i = 0$$

For pipe flow the friction factor is:

$$f \equiv \frac{\Delta p D}{2L\rho v^2} = \frac{16}{\mathrm{Re}} \text{ (laminar flow) } = \frac{16\mu}{\rho v D}$$

$$f = 0.079\,\mathrm{Re}^{-0.25} \text{ (turbulent flow)}$$

$$l_v = \frac{2v^2 Lf}{D} = \frac{2\langle v \rangle^2 Lf}{D}$$

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho \mathrm{v}_x}{\partial x} + \frac{\partial \rho \mathrm{v}_y}{\partial y} + \frac{\partial \rho \mathrm{v}_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r \mathrm{v}_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho \mathrm{v}_\theta)}{\partial \theta} + \frac{\partial(\rho \mathrm{v}_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 \mathrm{v}_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\rho \mathrm{v}_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial(\rho \mathrm{v}_\phi)}{\partial \phi}\right)$

TABLE 3.2

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial \mathrm{v}_x}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_x}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \tag{3.3.18a}$$

y component

$$\rho\left[\frac{\partial \mathrm{v}_y}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_y}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_y}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] \tag{3.3.18b}$$

z component

$$\rho\left[\frac{\partial \mathrm{v}_z}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_z}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_z}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] \tag{3.3.18c}$$

Cylindrical coordinates

r component

$$\rho\left[\frac{\partial \mathrm{v}_r}{\partial t} + \mathrm{v}_r\frac{\partial \mathrm{v}_r}{\partial r} + \frac{\mathrm{v}_\theta}{r}\frac{\partial \mathrm{v}_r}{\partial \theta} - \frac{\mathrm{v}_\theta^2}{r} + \mathrm{v}_z\frac{\partial \mathrm{v}_r}{\partial z}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{zr}}{\partial z}\right] \tag{3.3.19a}$$

θ component

$$\rho\left[\frac{\partial \mathrm{v}_\theta}{\partial t} + \mathrm{v}_r\frac{\partial \mathrm{v}_\theta}{\partial r} + \frac{\mathrm{v}_\theta}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r\mathrm{v}_\theta}{r} + \mathrm{v}_z\frac{\partial \mathrm{v}_\theta}{\partial z}\right] = \rho g_\theta - \frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right] \tag{3.3.19b}$$

z component

$$\rho\left[\frac{\partial \mathrm{v}_z}{\partial t} + \mathrm{v}_r\frac{\partial \mathrm{v}_z}{\partial r} + \frac{\mathrm{v}_\theta}{r}\frac{\partial \mathrm{v}_z}{\partial \theta} + \mathrm{v}_z\frac{\partial \mathrm{v}_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right] \tag{3.3.19c}$$

Spherical coordinates

r component

$$\rho\left[\frac{\partial \mathrm{v}_r}{\partial t} + \mathrm{v}_r\frac{\partial \mathrm{v}_r}{\partial r} + \frac{\mathrm{v}_\theta}{r}\frac{\partial \mathrm{v}_r}{\partial \theta} + \frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial \mathrm{v}_r}{\partial \phi} - \frac{\mathrm{v}_\theta{}^2 + \mathrm{v}_\phi{}^2}{r}\right] = \rho g_r - \frac{\partial p}{\partial r} + \left[\frac{1}{r^2}\frac{\partial (r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tau_{\theta r}\sin\theta)}{\partial \theta} \right.$$

$$\left. + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] \tag{3.3.20a}$$

TABLE 3.3

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{3.3.22b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \tag{3.3.22f}$$

Cylindrical coordinates

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \tag{3.3.23a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{3.3.23b}$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \tag{3.3.23c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_r}{r}\right) \tag{3.3.23d}$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \tag{3.3.23e}$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \tag{3.3.23f}$$

Spherical coordinates

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \tag{3.3.24a}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) \tag{3.3.24b}$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\right) \tag{3.3.24c}$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_r}{r}\right) \tag{3.3.24d}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \tag{3.3.24e}$$

$$\tau_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) \tag{3.3.24f}$$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x \qquad (3.3.26a)$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y \qquad (3.3.26b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad (3.3.26c)$$

Cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad (3.3.27a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad (3.3.27b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad (3.3.27c)$$