CBE 30357

9/7/17

TOPICS FOR TODAY

1) QUICK REVIEW OF DERIVATION OF MOMENTUM EQS

2) ...THEN TRANSFORMATION TO NAVIER-STOKES EQ'S

3) USE OF THESE TO SOLVE PROBLEMS

4) A LITTLE MORE ABOUT VISCOSITY

REVIEW: DERIVATION...

RATE OF CHANGE OF MOMENTUM IN C.V.

$$\frac{d\,\rho\vec{V}}{dt}\,\Delta x\Delta y\Delta z$$

= RATE OF MOMENTUM FLOW INTO C.V.

(E.G.) $\rightarrow \rho\vec{V}\;V_x\,\Delta y\Delta z\Big|_x$

− RATE OF MOMENTUM FLOW OUT OF C.V.

$-\rho\vec{V}\;V_x\,\Delta y\Delta z\Big|_{x+\Delta x}$

+ Σ FORCES

BODY → $\rho g\,\Delta x\Delta y\Delta z$

AREA

SURFACE F.C

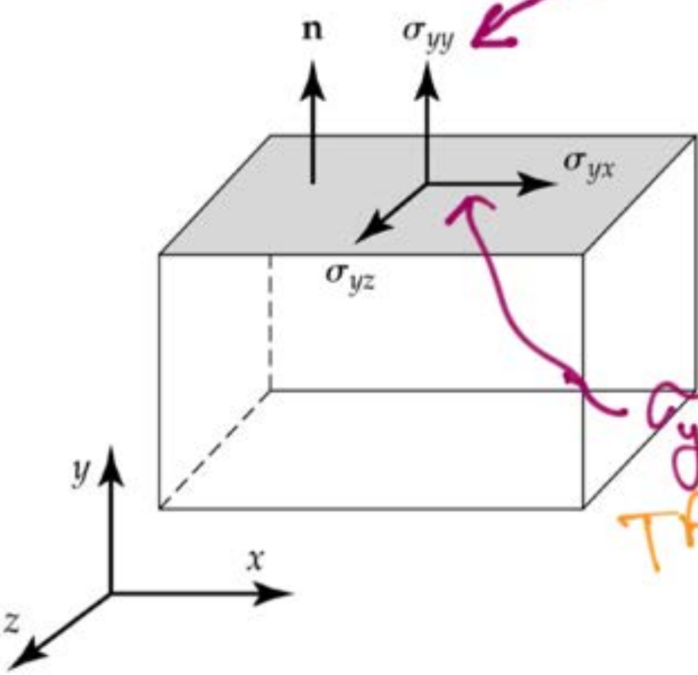
$\sigma_{xx}(\Delta y\Delta z)\Big|_x + \sigma_{yx}\,\Delta z\Delta x\Big|_y + \sigma_{zx}\,\Delta x\Delta y\Big|_z$

FORCE/AREA

REVIEW

SURFACE FORCES

Figure 2.9 Cubic fluid element showing the stresses acting on a face of constant y. As shown, the normal stress σ_{yy} is tensile.



σ_{yy} NORMAL FORCE

σ_{yx} TANGENTIAL FORCE

PEARSON
Transport Phenomena in Biological Systems, Second Edition
George A. Truskey, Fan Yuan, and David F. Katz

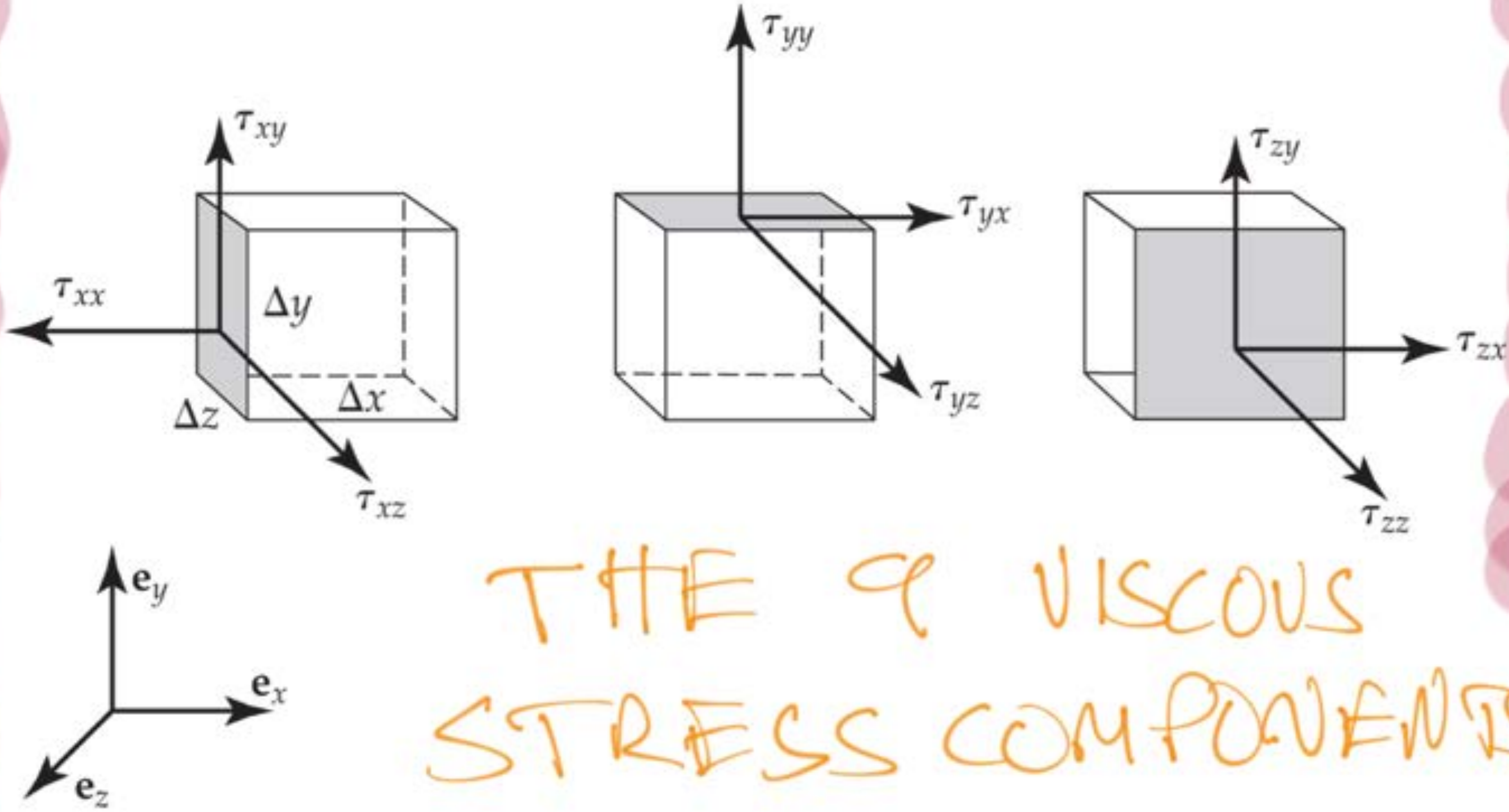
σ_{yy}

FACE (ORIENTATION)

DIRECTION (OF FORCE)

REVIEW

Figure 3.5 Schematic of components of the viscous stress tensor.



THE 9 VISCOUS STRESS COMPONENTS

$$\bar{\bar{\sigma}} = \bar{\bar{\tau}} - \bar{\bar{I}} p$$

REVIEW

COMPLETE EXPRESSION FOR SURFACE FORCES

$$\overline{\overline{\sigma}} = \overline{\overline{\tau}} - \overline{\overline{I}}\,P$$

PRESSURE

ALL SURFACE FORCES

VISCOUS SURFACE FORCES

$$\begin{Bmatrix} i = 1,2,3 \\ j = 1,2,3 \end{Bmatrix}$$

$$\overline{\overline{\sigma}} = \sigma_{ij} = \begin{pmatrix} \tau_{xx}-P & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy}-P & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz}-P \end{pmatrix}$$

$$\begin{Bmatrix} x \to 1 \\ y \to 2 \\ z = 3 \end{Bmatrix}$$

TANGENTIAL COMPONENTS

NORMAL COMPONENTS

REVIEW

$$dV\left(\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v}\right) =$$

$$dV\left(-\vec{\nabla} P + \vec{\nabla} \cdot \bar{\bar{\tau}} + \rho \vec{g}\right)$$

WHICH GIVES...

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} P + \vec{\nabla} \cdot \bar{\bar{\tau}} + \rho \vec{g}$$

CAUCHY - MOMENTUM EQS

REVIEW

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

y component

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

"CAUCHY-MOMENTUM EQUATIONS"

THESE ARE GENERALLY APPLICABLE BUT THERE ARE TOO MANY UNKNOWNS

NEED CONSTITUTIVE EQUATION TO RELATE τ'S TO $\nabla \vec{v}$

E.G. $\tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$

REVIEW

SIMPLEST VALID RELATION IS CALLED: "NEWTONIAN FLUID"

$i = 1,2,3$
$j = 1,2,3$

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

SYMMETRY $\tau_{ji} = \tau_{ij}$ [ALWAYS]

$\mu > 0$ [FROM "2ND LAW"]

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{3.3.22b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \tag{3.3.22f}$$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

$$\frac{d\bar{P}}{dt} = \Sigma \bar{F}_i$$

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

TABLE 3.1

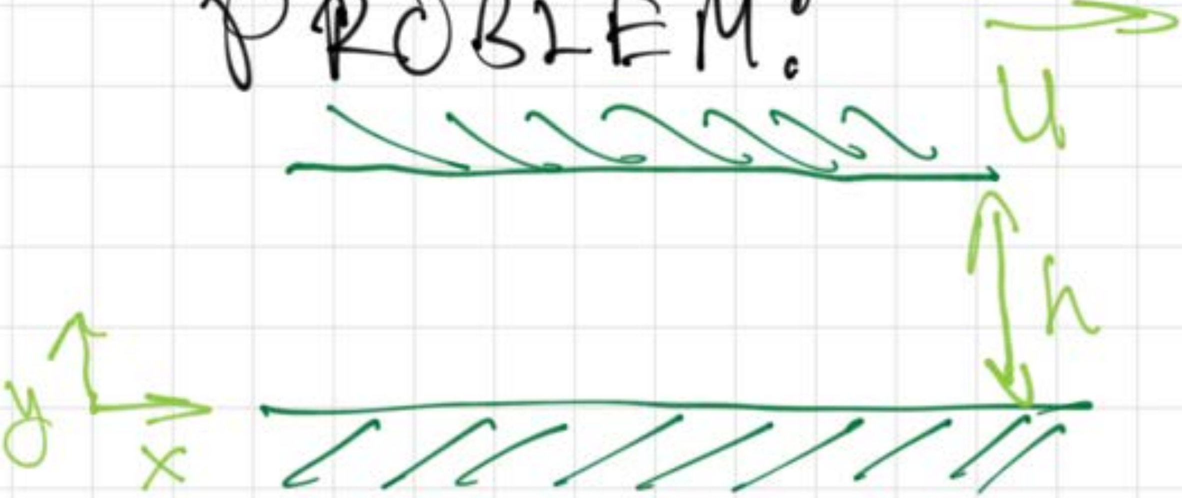
The Conservation of Mass (Continuity Equation)

Rectangular coordinates (x, y, z) $\quad \frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$

These equations can be used to solve almost any fluid flow problem if the fluid is Newtonian.

Our challenge is to learn to use them!

WE START WITH THE SIMPLEST PROBLEM:



BOTTOM PLATE IS FIXED

TOP PLATE MOVES AT U

WHAT IS $V_x(y)$?

WHAT FORCE IS REQUIRED?

NEWTONIAN FLUID, μ

ρ = CONST

LOOK AT CONTINUITY

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

BUT $\frac{\partial}{\partial z} = 0$

NOTHING CHANGES IN X

$\frac{\partial}{\partial x} = 0 \quad \therefore \frac{\partial v_y}{\partial y} = 0$

ONLY $V_x(y)$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y \frac{\partial v_x}{\partial y}} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial x}} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \cancel{\rho g_x}$$

y direction

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x \cancel{\frac{\partial v_y}{\partial x}} + v_y \cancel{\frac{\partial v_y}{\partial y}} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho g_y$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_x \cancel{\frac{\partial v_z}{\partial x}} + \cancel{v_y} \cancel{\frac{\partial v_z}{\partial y}} + \cancel{v_z} \cancel{\frac{\partial v_z}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial z}} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \cancel{\rho g_z}$$

AFTER SOME WORK

$$0 = \mu \frac{\partial^2 v_x}{\partial y^2}$$

BOUNDARY CONDITIONS..

$$V_x(0) = 0, \quad V_x(h) = U$$

FLUID STICKS TO SOLID SURFACES!

SOLVE

$$0 = \mu \frac{d}{dy}\left(\frac{dv_x}{dy}\right)$$

$$\int d\left(\frac{dv_x}{dy}\right) = \int 0$$

$$\frac{dv_x}{dy} - C_1 = 0$$

$$\int d(v_x) = \int C_1 \, dy$$

$$v_x = C_1 y + C_2$$

FIT B.C.'S

$V_x(0) = 0 \quad \therefore C_2 = 0$

$V_x(h) = U = C_1 h$

$\therefore C_1 = \frac{U}{h}$

$$\therefore \boxed{V_x(y) = \frac{y}{h} U}$$

LINEAR PROFILE

μ DIVIDED OUT AND SO DOES NOT AFFECT PROFILE.

WHICH STRESS COMPONENTS ARE IMPORTANT ??

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \cancel{\frac{\partial v_x}{\partial x}}\ 0 \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}\ 0\right) \tag{3.3.22b}$$

$$\tau_{yx} = \mu \frac{\partial v_x}{\partial y}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\cancel{\frac{\partial v_x}{\partial z}}\ 0 + \cancel{\frac{\partial v_z}{\partial x}}\ 0\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \cancel{\frac{\partial v_y}{\partial y}}\ 0 \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\cancel{\frac{\partial v_y}{\partial z}}\ 0 + \cancel{\frac{\partial v_z}{\partial y}}\ 0\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \cancel{\frac{\partial v_z}{\partial z}} \tag{3.3.22f}$$

$$v_x(y) = \frac{y}{h} U$$ ← NO VISCOSITY

CONSTANT, NOT A FUNCTION OF y

$$\tau_{yx} = \mu \frac{\partial v_x}{\partial y} = \frac{\mu U}{h}$$ ← VISCOSITY IN STRESS

If the question is: how much force will it take to push the plate at a given velocity ?

You want: force/area * area

(FORCE/AREA)

$$\text{FORCE}_x = \tau_{yx} \, W \times L$$

WIDTH IN Z

LENGTH IN X

FASTER ↑ F

$$= \mu \frac{u}{h} W L$$

MORE VISCOUS ↑ F

CLOSER SPACING ↑ F

It is obvious, but what is the average velocity of the fluid?

$$\text{AVERAGE VELOCITY} = \frac{\text{TOTAL VOLUMETRIC FLOW}}{\text{AREA OF FLOW}}$$

$$\langle V_x \rangle = \frac{\int_0^w \int_0^h V_x(y)\,dy\,dz}{\int_0^w \int_0^h dy\,dz}$$

$$= \frac{\int_0^w dz \int_0^h \frac{u y}{h}\,dy}{w\,h}$$

$$\langle V_x \rangle = \frac{w \frac{u}{h} \frac{y^2}{2}\Big|_0^h}{w\,h} = \frac{u}{2}$$

SUPPOSE WE KEEP THE FLOW GEOMETRY THE SAME, BUT CAUSE THE FLOW BY A PRESSURE GRADIENT

$P_1 > P_2$

NO CHANGE W/Z

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x \frac{\partial v_x}{\partial x}} + \cancel{v_y \frac{\partial v_x}{\partial y}} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \cancel{\rho g_x}$$

y direction

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x \frac{\partial v_y}{\partial x}} + v_y \frac{\partial v_y}{\partial y} + \cancel{v_z \frac{\partial v_y}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial y}} + \mu\left[\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \cancel{\rho g_y}$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x \frac{\partial v_z}{\partial x}} + \cancel{v_y \frac{\partial v_z}{\partial y}} + \cancel{v_z \frac{\partial v_z}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial z}} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \cancel{\rho g_z}$$

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 V_x}{\partial y^2}$$

BOUNDARY CONDITIONS:

$$V_x(h) = 0$$
$$V_x(-h) = 0$$

[NO SLIP]

WE SOLVE:

$$\frac{d^2 V_x}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$$

A subtle and perhaps confusing argument will now be given! 😳

We conclude that V_x does not change in x.

We conclude that $\frac{\partial P}{\partial x}$ does not change with y

Hence, the only way for these terms to be always equal is if they are equal to a constant.

$$\frac{d^2 V_x}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} = \text{CONSTANT}$$

NOW WE PROCEED...

$$\frac{d}{dy}\left(\frac{dV_x}{dy}\right) = \frac{1}{\mu}\frac{dP}{dx}$$

$$d\left(\frac{dV_x}{dy}\right) = \frac{1}{\mu}\frac{dP}{dx}\,dy$$

$$\int d\left(\frac{dV_x}{dy}\right) = \int \frac{1}{\mu}\frac{dP}{dx}\,dy$$

$$\frac{dv_x}{dy} = \frac{1}{\mu}\frac{dP}{dx}y + C_1$$

$$dv_x = \left(\frac{1}{\mu}\frac{dP}{dx}y + C_1\right)dy$$

$$\int dv_x = \int\left(\frac{1}{\mu}\frac{dP}{dx}y + C_1\right)dy$$

$$v_x = \frac{1}{\mu}\frac{dP}{dx}\frac{y^2}{2} + C_1 y + C_2$$

NOW WE NEED TO APPLY BOUNDARY CONDITIONS

$$v_x(h) = 0 = \frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2} + C_1 h + C_2$$

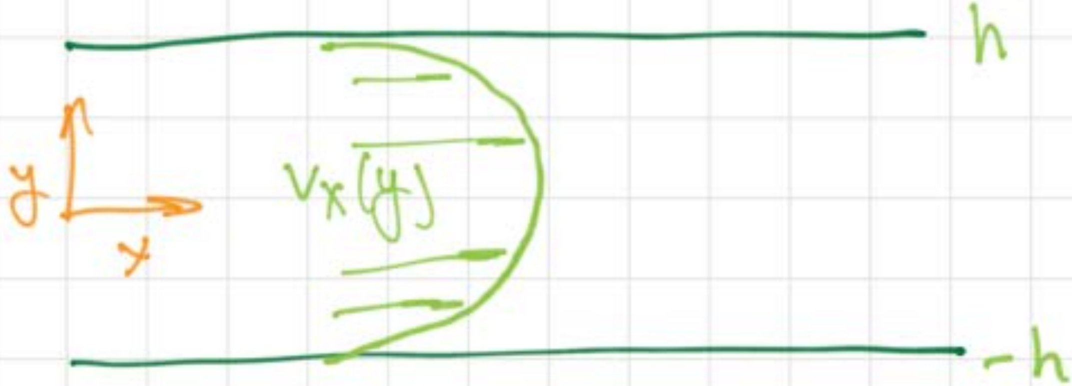
$$v_x(-h) = 0 = \frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2} - C_1 h + C_2$$

$$\therefore C_1 = 0$$

$$C_2 = -\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}$$

WHICH GIVES

$$v_x(y) = -\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}\left(1 - \frac{y^2}{h^2}\right)$$

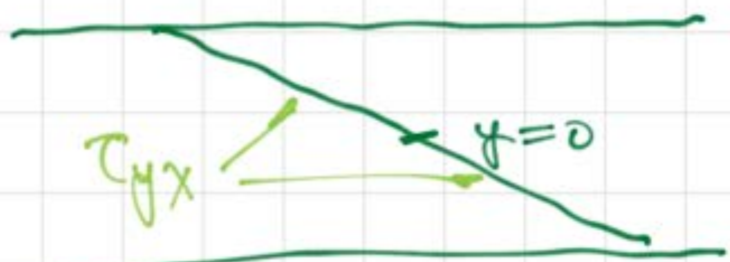


SHEAR STRESS

$$\tau_{yx} = \mu \frac{\partial v_x}{\partial y}$$

$$= \frac{dP}{dx} y$$

(−)



STRESS IS MAXIMUM AT WALLS, 0 IN MIDDLE

WHAT IS AVERAGE VELOCITY?

$$\langle v_x \rangle = \frac{\int_0^w \int_{-h}^{h} v_x(y)\, dy\, dz}{\int_0^w \int_{-h}^{h} dy\, dz}$$

$$v_x(y) = -\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}\left(1 - \frac{y^2}{h^2}\right)$$

$$\langle v_x \rangle = \frac{-\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}w}{2wh}\int_{-h}^{h}\left(1 - \frac{y^2}{h^2}\right)dy$$

$$= \frac{-\frac{1}{\mu}\frac{dP}{dx}\frac{h^{\cancel{2}}\cancel{w}}{2}}{2\cancel{w}\cancel{h}}\left(y - \frac{y^3}{3h^2}\right)\Bigg|_{-h}^{h}$$

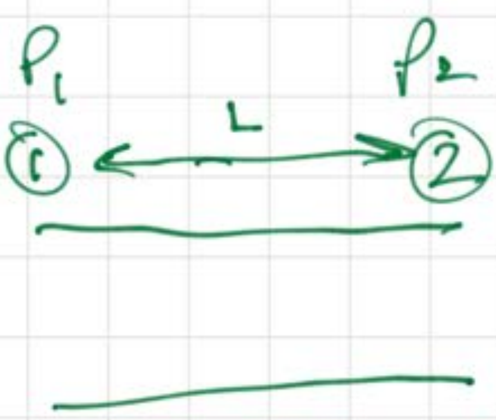
$$= \frac{-1}{4\mu}\frac{dP}{dx}\, h\left(\underbrace{\left(h - \frac{1}{3}h\right)}_{2/3\,h} - \underbrace{\left(-h + \frac{1}{3}h\right)}_{+2/3\,h}\right)$$

$$\langle v_x \rangle = \frac{1}{3\mu}\left(-\frac{dP}{dx}\right) h^2$$

PRESSURE "DROPS" LINEARLY WITH DISTANCE, HENCE

$$\frac{dP}{dx} < 0$$

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} = \text{CONSTANT}$$



$$\frac{dP}{dx} = C_1$$

$$dP = C_1 dx$$

$$\int dP = \int C_1 dx$$

$$P = C_1 x + C_2$$

@ $x=0$, $P=P_1$

$$\therefore \quad P = C_1 x + P_1$$

IF $P = P_2$ @ $x = L$

$$P_2 = C_1 L + P_1$$

$$C_1 = \frac{P_2 - P_1}{L}$$

OR

$$P = \frac{P_2 - P_1}{L} x + P_1$$

$$= \frac{dP}{dx} < 0$$

PRESSURE DECREASES LINEARLY WITH DISTANCE

How does gravity cause fluid flow?

X- DIRECTION NAVIER-STOKES EQ.

$$0 = \mu \frac{d^2 V_x}{dy^2} + \rho g \cos\theta$$

<u>BOUNDARY CONDITIONS</u>

$$V_x(0) = 0$$

AT $y = h$ WE HAVE TO FIND SOMETHING ELSE....

$$\tau_{yx}(y=h) = 0$$

NO <u>STRESS</u> ☎

Boundary conditions:

1. Fluid sticks to solid surfaces $V_x(0) = 0$
2. Fluid sticks to another immiscible fluid $V_x^{I}(h) = V_x^{II}(h)$
3. The shear stress is continuous across the fluid-fluid interface

$$\tau_{yx}^{I}(h) = \tau_{yx}^{II}(h)$$

$$\mu^{I} \frac{\partial V_x^{I}}{\partial y}\bigg|_h = \mu^{II} \frac{\partial V_x^{II}}{\partial y}$$

4. For liquid flowing next to quiescent air, the stress can be considered approximately 0.

$$\tau_{yx}(h) \approx 0$$

$$\frac{\partial V_x}{\partial y}(h) \approx 0$$

"FALLING FILM"

$$0 = \mu \frac{d^2 v_x}{dy^2} + \rho g \cos\theta$$

$$\frac{d^2 v_x}{dy^2} = -\frac{\rho g}{\mu} \cos\theta$$

$$\frac{d}{dy}\left(\frac{dv_x}{dy}\right) = -\frac{\rho g}{\mu} \cos\theta$$

$$\int d\left(\frac{dv_x}{dy}\right) = -\frac{\rho g}{\mu} \cos\theta \int dy$$

$$\frac{dv_x}{dy} = -\frac{\rho g}{\mu} \cos\theta \, y + C_1$$

$$\int dv_x = \int\left(-\frac{\rho g}{\mu} \cos\theta \, y + C_1\right) dy$$

$$v_x = -\frac{\rho g}{\mu} \cos\theta \frac{y^2}{2} + C_1 y + C_2$$

@ $y=0$ $v_x(0)=0$

$\therefore C_2 = 0$

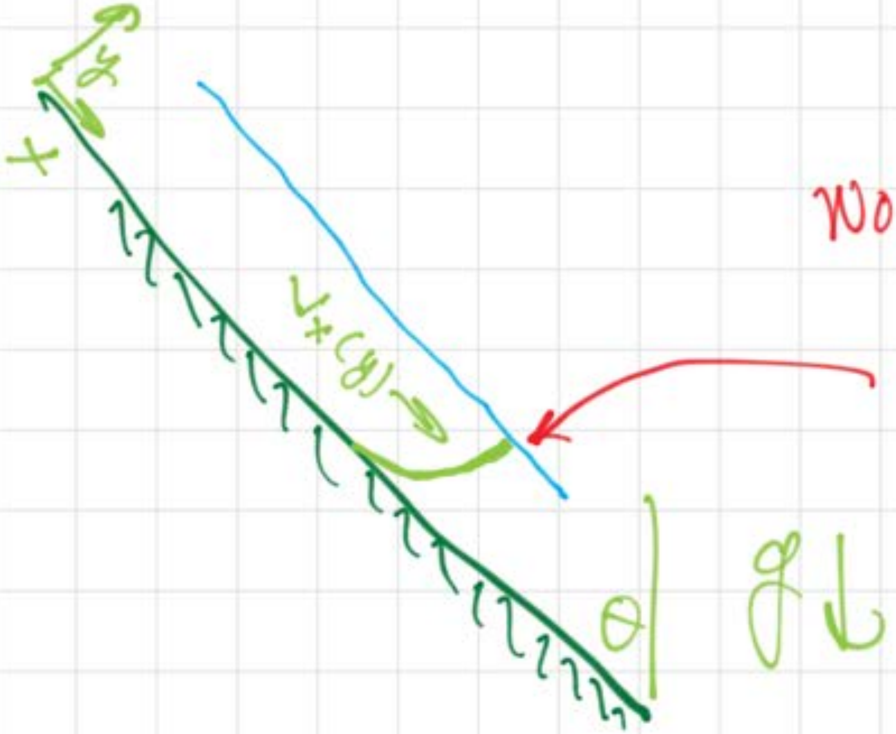
@ $y=h$ $\frac{dv_x}{dy}=0$

$$\frac{dv_x}{dy} = -\frac{\rho g}{\mu}\cos\theta\, y + C_1$$

$$\therefore C_1 = \frac{\rho g}{\mu}\cos\theta\, h$$

THUS THE PROFILE IS

$$v_x(y) = \frac{\rho g}{\mu}\cos\theta\left(\frac{y^2}{2} - yh\right)$$



NO STRESS

$$\left.\frac{dv_x}{dy}\right|_{y=0} = h$$