CBE 30357

AUGUST 31, 2017

DERIVATION OF DIFFERENTIAL MASS AND MOMENTUM EQUATIONS

THESE EQUATIONS PROVIDE A WAY TO SOLVE ~ALL FLUID FLOW PROBLEMS ON ANY LENGTH SCALE

WHILE THE STEPS OF THE DERIVATION ARE NOT ESSENTIAL TO THEIR USE, GOING THROUGH THESE IN DETAIL PROVIDES INSIGHT INTO THE PHYSICS ASSOCIATED WITH EACH TERM

THIS AIDS IN SOLVING PROBLEMS AND UNDERSTANDING RESULTS...

RECALL A QUESTION FROM LAST TIME...

FLOW IN TUBE

What is relation between ΔP and velocity?

? USE MASS BALANCE ?

Table 2.2-1 The Mass Conservation Equation

	Mass Basis	Molar Basis
Rate-of-change form of the mass balance		
General equation	$\frac{dM}{dt} = \sum_{k=1}^{K} \dot{M}_k$	$\frac{dN}{dt} = \sum_{k=1}^{K} \dot{N}_k$
Special case:		
Closed system	$\frac{dM}{dt} = 0$ $M = \text{constant}$	$\frac{dN}{dt} = 0$ $N = \text{constant}$
*Difference form of the mass balance**		
General equation	$M_2 - M_1 = \sum_{k=1}^{K} \Delta M_k$	$N_2 - N_1 = \sum_{k=1}^{K} \Delta N_k$
Special cases:		
Closed system	$M_2 = M_1$	$N_2 = N_1$
Steady flow	$M_2 - M_1 = \sum_{k=1}^{K} \dot{M}_k \, \Delta t$	$N_2 - N_1 = \sum_{k=1}^{K} \dot{N}_k \, \Delta t$

*Here we have used the abbreviated notation $M_i = M(t_i)$ and $N_i = N(t_i)$.

water leaves the tank by evaporation. How much water is in the tank at the end of the [illegible] period?

$$\cancel{\frac{dM}{dt}}^{\,0} = \dot{M}_{IN} - \dot{M}_{OUT}$$

$$\dot{M}_{IN} = \dot{M}_{OUT}$$

$$\rho_1 \langle v \rangle_1 A_1 = \rho_2 \langle v \rangle_2 A_2$$

BUT: $\rho_1 = \rho_2$

$A_1 = A_2$

$\therefore \; \langle v \rangle_1 = \langle v \rangle_2$

TRUE, BUT NOT USEFUL!!

OK. HOW ABOUT ENERGY BALANCE!!

$v_1 = v_2$
$V_1 = V_2$

$$\underbrace{\frac{d}{dt}\left[U + m\left(\frac{u^2}{2} + gz\right)\right]}_{\to 0} = \dot{Q} + \underbrace{\cancel{\dot{W}}}_{\to 0} + \sum_{i=1}^{k} \dot{m}_i \left(\underline{H}_i + \underbrace{\cancel{\frac{u_i^2}{2}}}_{\to 0} + \underbrace{\cancel{gz_i}}_{\to 0}\right)$$

COULD MAKE $\dot{Q} = 0$

$\underline{H}_1 = \underline{H}_2$

$\underline{H} \equiv \underline{U} + P\underline{V}$

$P_1 \neq P_2$, T MUST CHANGE?

(OR $\dot{Q}$ IS SLIGHTY NEGATIVE...)

WE ARE NOT MAKING MUCH PROGRESS. !!

WE WILL NEED ANOTHER PHYSICAL LAW:

CONSERVATION OF MOMENTUM !!

To proceed we need a formalism that can describe mass, momentum and energy transport and can be adapted to all problems of interest.

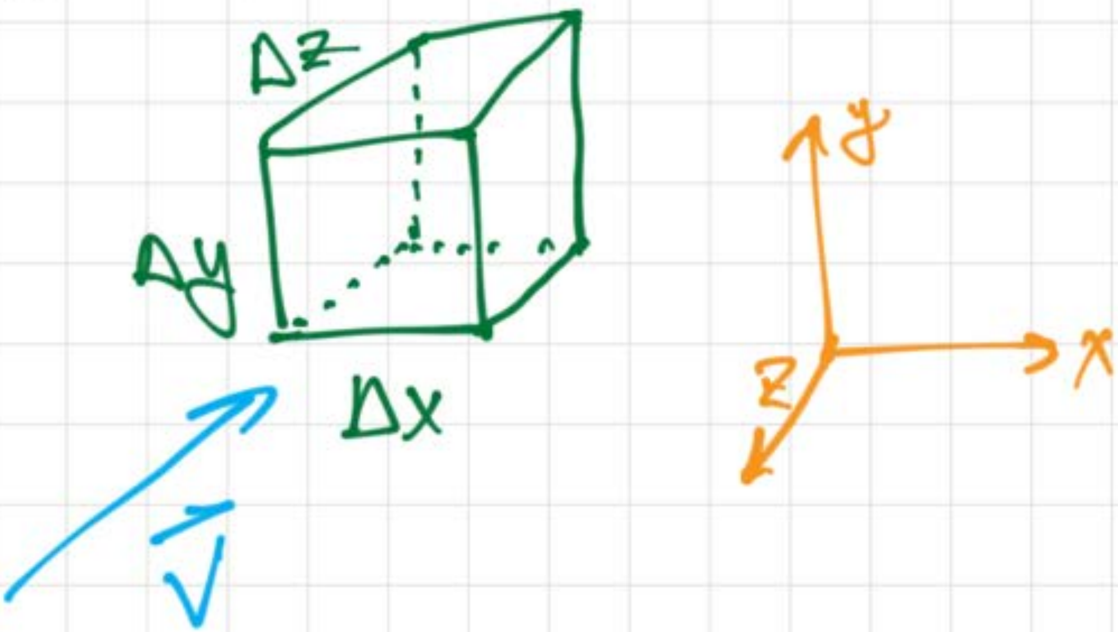
This formalism is the set of partial differential equations that I showed before.

We will start by explaining where these come from.

Then, we will use them!

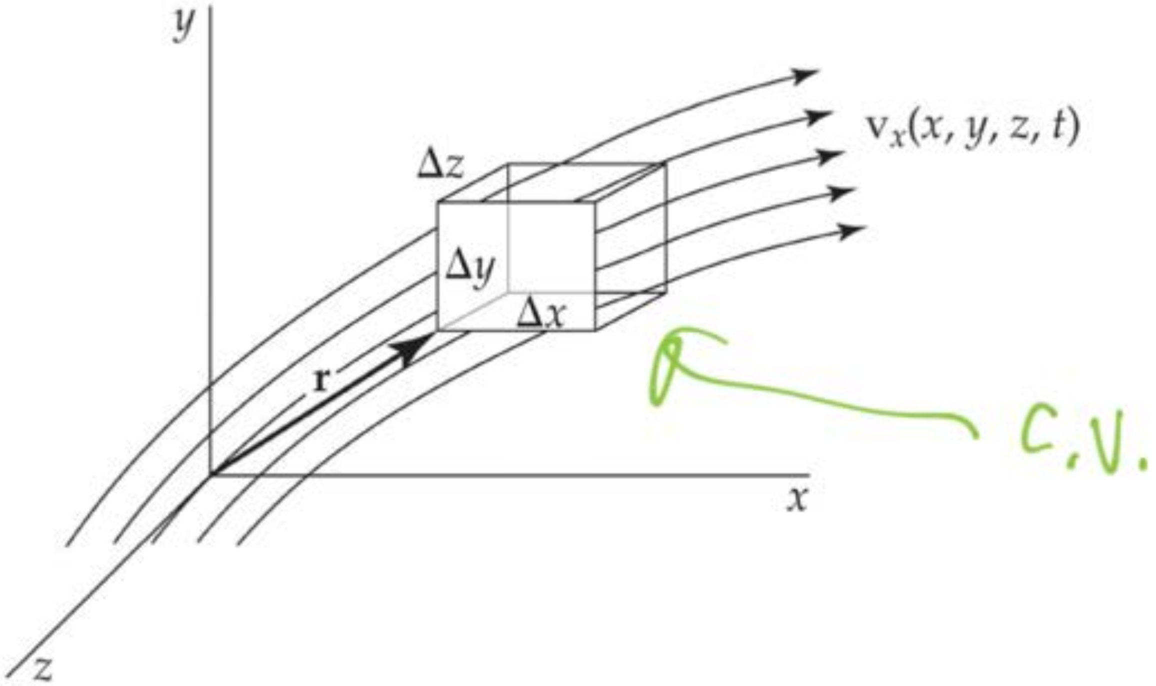
Mass conservation for a single component is the simplest equation, so we do it first.

WE WILL START WITH A "CONTROL VOLUME" THAT IS OF DIFFERENTIAL SIZE



$$\vec{V} = V_x \hat{e}_x + V_y \hat{e}_y + V_z \hat{e}_z$$

Figure 2.3 Motion of fluid through a volume $\Delta x\ \Delta y\ \Delta z$.



CONSERVATION PRINCIPLE:

RATE OF CHANGE OF MASS IN C.V.
= FLOW RATE OF MASS INTO C.V
− FLOW RATE OF MASS OUT
OF C.V.

WHAT IS RATE OF CHANGE OF MASS IN C.V.?

$$\frac{d}{dt} M = \frac{d}{dt}(\rho \Delta x \Delta y \Delta z)$$

ONLY $\rho = \rho(t)$ $\therefore$

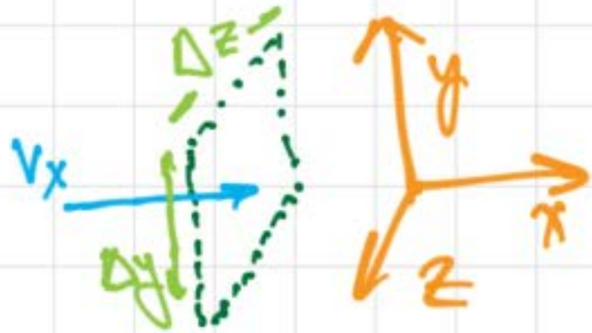
$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

HOW ABOUT AN "INFLOW"?

Need a way to keep track of any flow situation

Thus we consider each face separately and allow 1 in_flow and 1 out_flow for each coordinate direction.

v_x will carry fluid in or out across "x" faces



MASS FLOW ACROSS X-FACE

$$\frac{\text{CONSERVED QUANTITY}}{\text{VOLUME}} \times \text{VOLUMETRIC FLOW RATE}$$

$$\rho \quad \times \quad V_x \Delta y \Delta z$$

SO FOR THE JUST THE X- DIRECTION:

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \left(\rho V_x \big|_x - \rho V_x \big|_{x+\Delta x} \right) \Delta y \Delta z$$

DIVIDE BY $\Delta x \Delta y \Delta z$

$$\frac{\partial \rho}{\partial t} = - \frac{\left(\rho V_x \big|_{x+\Delta x} - \rho V_x \big|_x \right)}{\Delta x}$$

LIMIT $\Delta x \Rightarrow 0$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} \rho V_x$$

$$= -V_x \frac{\partial \rho}{\partial x} - \rho \frac{\partial V_x}{\partial x}$$

CONSIDER SIMPLE FLOW IN 1 DIMENSION

IF ρ = CONST

$$\frac{\partial v_x}{\partial x} = 0$$

V_x IS NOT A FUNCTION OF x

NOW EXTEND DERIVATION TO ALL 3 SPATIAL DIMENSIONS

MASS CONSERVATION:

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} =$$

X-FACES $\left(\rho V_x \big|_{x} - \rho V_x \big|_{x+\Delta x} \right) \Delta y \Delta z$

$+$

Y-FACES $\left(\rho V_y \big|_{y} - \rho V_y \big|_{y+\Delta y} \right) \Delta z \Delta x$

$+$

Z-FACES $\left(\rho V_z \big|_{z} - \rho V_z \big|_{z+\Delta z} \right) \Delta x \Delta y$

DIVIDE BY $\Delta x \Delta y \Delta z$

TAKE LIMIT AS $\Delta x, \Delta y, \Delta z \to 0$

E.G.: $-\left(\frac{\rho V_y|_{y+\Delta y} - \rho V_y|_y}{\Delta y}\right)$

LIMIT AS $\Delta y \to 0$

$$= -\frac{\partial \rho V_y}{\partial y}$$

FULL EQ:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho V_x)}{\partial x} - \frac{\partial(\rho V_y)}{\partial y} - \frac{\partial(\rho V_z)}{\partial z}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} = 0}$$

IF $\rho = \text{CONST}$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\boxed{\vec{\nabla} \cdot \vec{v} = 0}$$

CONTINUITY EQUATION

EXAMPLE USE:

ENTRANCE SECTION

NOT 0 $\quad \frac{\partial V_x}{\partial x} = -\frac{\partial V_y}{\partial y}$ ← THEN THIS IS NOT 0

$\therefore V_y \neq 0$

Now consider conservation of momentum.

The derivation and the resulting equation will be more complicated than mass because "forces" act on the fluid to change the momentum.

In addition, the forces and velocity are vectors.

$$\frac{d\vec{P}}{dt} = m\vec{a} = \Sigma \vec{F}$$

WE WILL APPLY THIS "LAW" TO A DIFFERENTIAL CONTROL VOLUME.

BEFORE WE GET COMPLETELY "BOGGED-DOWN" IN THE DERIVATION, IT IS USEFUL TO NOTE:

1) WE WILL WRITE FINISHED EQ AS:

ACCELERATION = SUM OF FORCES

2) MANY INTERESTING FLOWS HAVE NO ACCELERATION

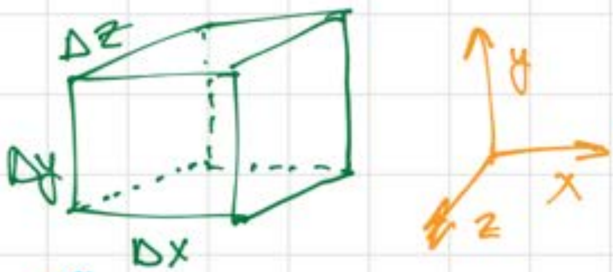
$\Sigma F_i = 0$ NO CHANGE IN TIME, NO CHANGE OF DIRECTION

3) FOR THIS CASE: TWO FORCES BALANCE

Newton's second law may be most tangible for discrete solid objects (e.g. a billiard ball), but it also works for "fluid particles".

The conserved quality for our equation formalism will be momentum per volume:

$$\frac{``m\vec{V}"}{\text{VOLUME}} \Longrightarrow \rho\vec{V}$$

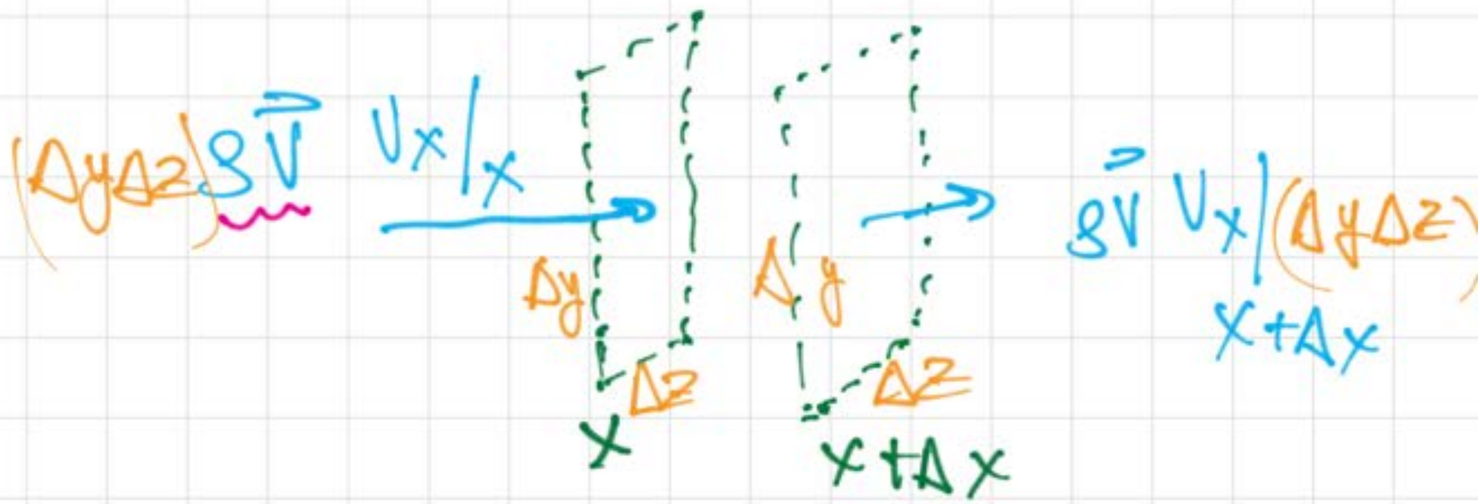


RATE OF CHANGE OF MOMENTUM IN CONTROL VOLUME:

$$\vec{V} = V_x\hat{e}_x + V_y\hat{e}_y + V_z\hat{e}_z$$

$$\frac{d}{dt}\left(\rho\vec{V}\,\Delta x\Delta y\Delta z\right) = \Delta x\Delta y\Delta z\,\frac{\partial(\rho\vec{V})}{\partial t}$$

WE NEXT NEED TO GET TERMS FOR FLOW OF MOMENTUM ACROSS THE FACES . . .



X - INFLOW:

$$\underbrace{\rho \vec{v}}_{\text{MOMENTUM/VOLUME}}\ \overbrace{v_x \Delta y \Delta z}^{\text{VOLUMETRIC FLOW}}\Big|_x$$

X - OUTFLOW: $\rho \vec{v}\ v_x \Delta y \Delta z \Big|_{x+\Delta x}$

$y \Rightarrow$

$$\rho\vec{v}\, v_y\, \Delta z \Delta x\Big|_{y} - \rho\vec{v}\, v_y\, \Delta z \Delta x\Big|_{y+\Delta y}$$

$z \Rightarrow$

$$\rho\vec{v}\, v_z\, \Delta x \Delta y\Big|_{z} - \rho\vec{v}\, v_z\, \Delta x \Delta y\Big|_{z+\Delta z}$$

SUM ALL TERMS

$$\Delta x \Delta y \Delta z \frac{\partial}{\partial t}\rho\vec{v} = -\Delta x \Delta y \Delta z \left[\frac{\left(\rho\vec{v}\, v_x\big|_{x+\Delta x} - \rho\vec{v}\, v_x\big|_{x}\right)}{\Delta x} + \frac{\rho\vec{v}\, v_y\big|_{y+\Delta y} - \rho\vec{v}\, v_y\big|_{y}}{\Delta y} + \frac{\rho\vec{v}\, v_z\big|_{z+\Delta z} - \rho\vec{v}\, v_z\big|_{z}}{\Delta z} \right]$$

THE REMAINING TERM
MUST REPRESENT
THE FORCES, SO
WE FIRST WRITE

$$" + \Sigma \vec{F} "$$

WE AGAIN WILL TAKE LIMIT AS
$\Delta x, \Delta y, \Delta z \to 0$

BUT WE WRITE: VOLUME

$$\lim_{\Delta x \Delta y \Delta z \to 0} = dV$$

(DON'T DIVIDE THROUGH YET)

THIS GIVES FOR OUR ENTIRE EQ.

$$dV\left(\frac{\partial}{\partial t}(\rho\vec{v})\right) = -dV\left(\frac{\partial}{\partial x}v_x(\rho\vec{v}) + \frac{\partial}{\partial y}v_y(\rho\vec{v}) + \frac{\partial}{\partial z}v_z(\rho\vec{v})\right) + \Sigma\vec{F}$$

$$\ldots = \vec{\nabla}\cdot(\rho\vec{v}\,\vec{v})$$ VECTOR SHORTHAND

FINALLY:

$$dV\left[\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\rho\vec{v}\,\vec{v})\right] = \Sigma\vec{F}$$

IF WE USE CONTINUITY EQ

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

THERE IS A SIMPLIFICATION:

$$dV \left[\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} \right] = \sum \vec{F}$$

To use one of my favorite expressions, the left side: "is what it is!"

We now must turn our attention to the forces.

Forces:

1) Body Forces. E.g., Gravity $\vec{E}, \vec{B}$

2) Surface forces... Pressure and shear stress.

BODY FORCE:

$\frac{\text{FORCE}}{\text{VOLUME}} \times \text{VOLUME}$

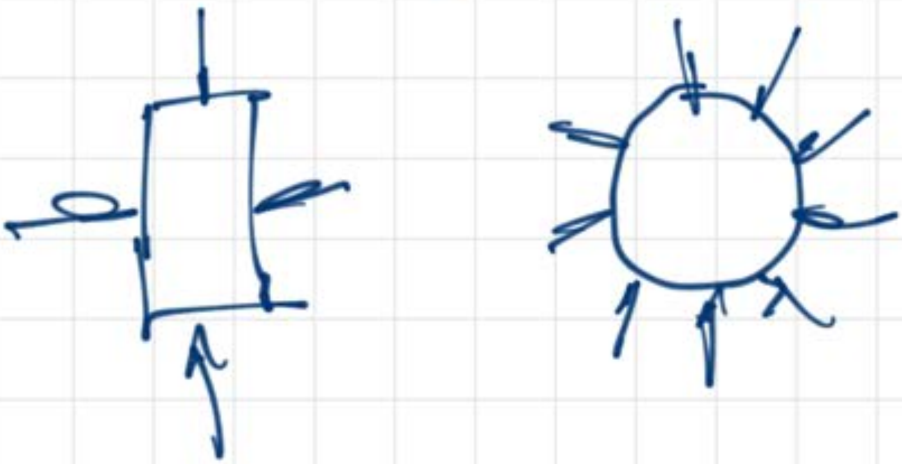
$\rho \vec{g} \quad \Delta x \Delta y \Delta z$

SURFACE FORCES

PRESSURE IS ISOTROPIC:
ACTS NORMAL TO SURFACE

TO DISCERN EFFECT: NEED ORIENTATION OF SURFACE

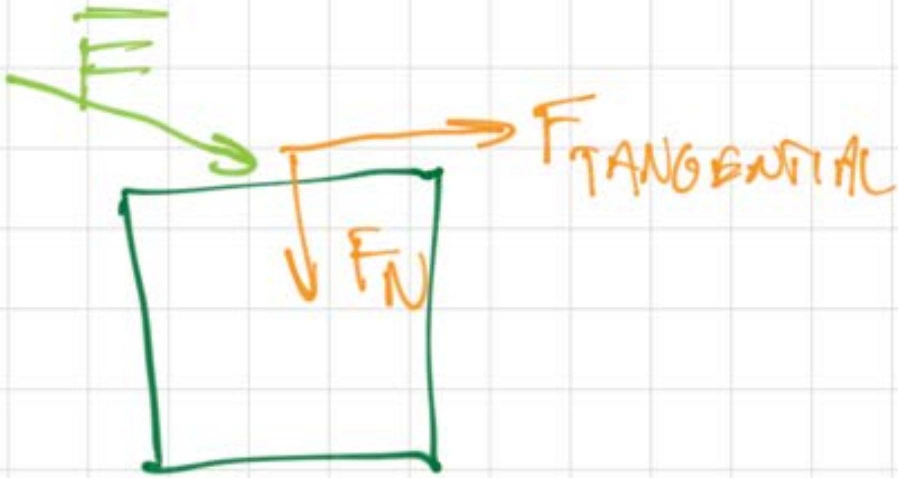
p



FOR AN ARBITRARY FORCE

PRESSURE + SHEAR

NORMAL + TANGENTIAL



BUT FOR EXACTLY
THE SAME FORCE

$\bar{F}_{TANGENTIAL} = 0$

THIS HAS ALL BEEN
FIGURED OUT

$$\bar{\bar{\sigma}} = \bar{\bar{\tau}} - \bar{\bar{I}} p$$

STRESS TENSOR

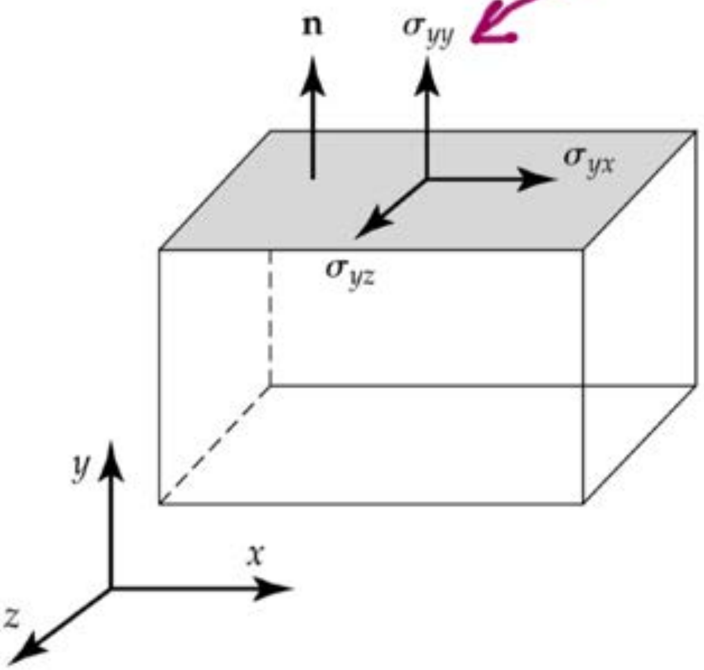
VISCOUS STRESSES

PRESSURE

SURFACE FORCES

NORMAL TO SURFACE

Figure 2.9 Cubic fluid element showing the stresses acting on a face of constant y. As shown, the normal stress σ_{yy} is tensile.



σ_{yy}

NORMAL FORCE

PEARSON *Transport Phenomena in Biological Systems*, Second Edition
George A. Truskey, Fan Yuan, and David F. Katz

σ_{yy}

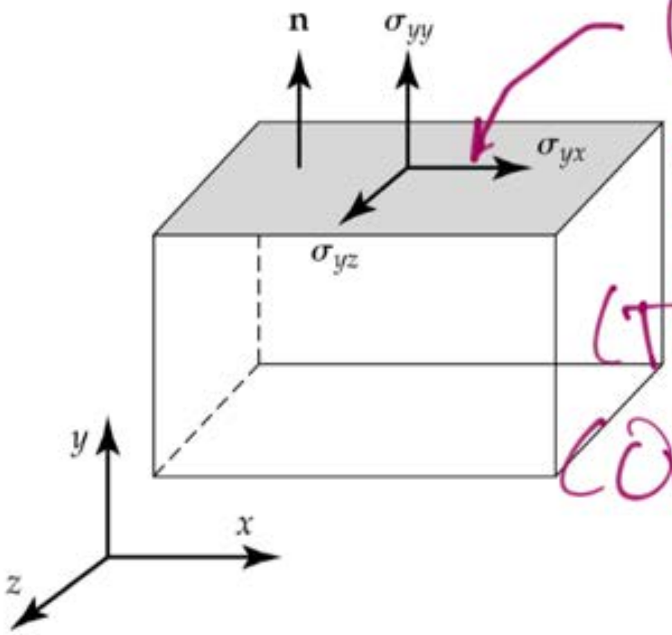
FACE (ORIENTATION)

DIRECTION (OF FORCE)

SURFACE FORCES

TANGENTIAL TO SURFACE

Figure 2.9 Cubic fluid element showing the stresses acting on a face of constant y. As shown, the normal stress σ_{yy} is tensile.



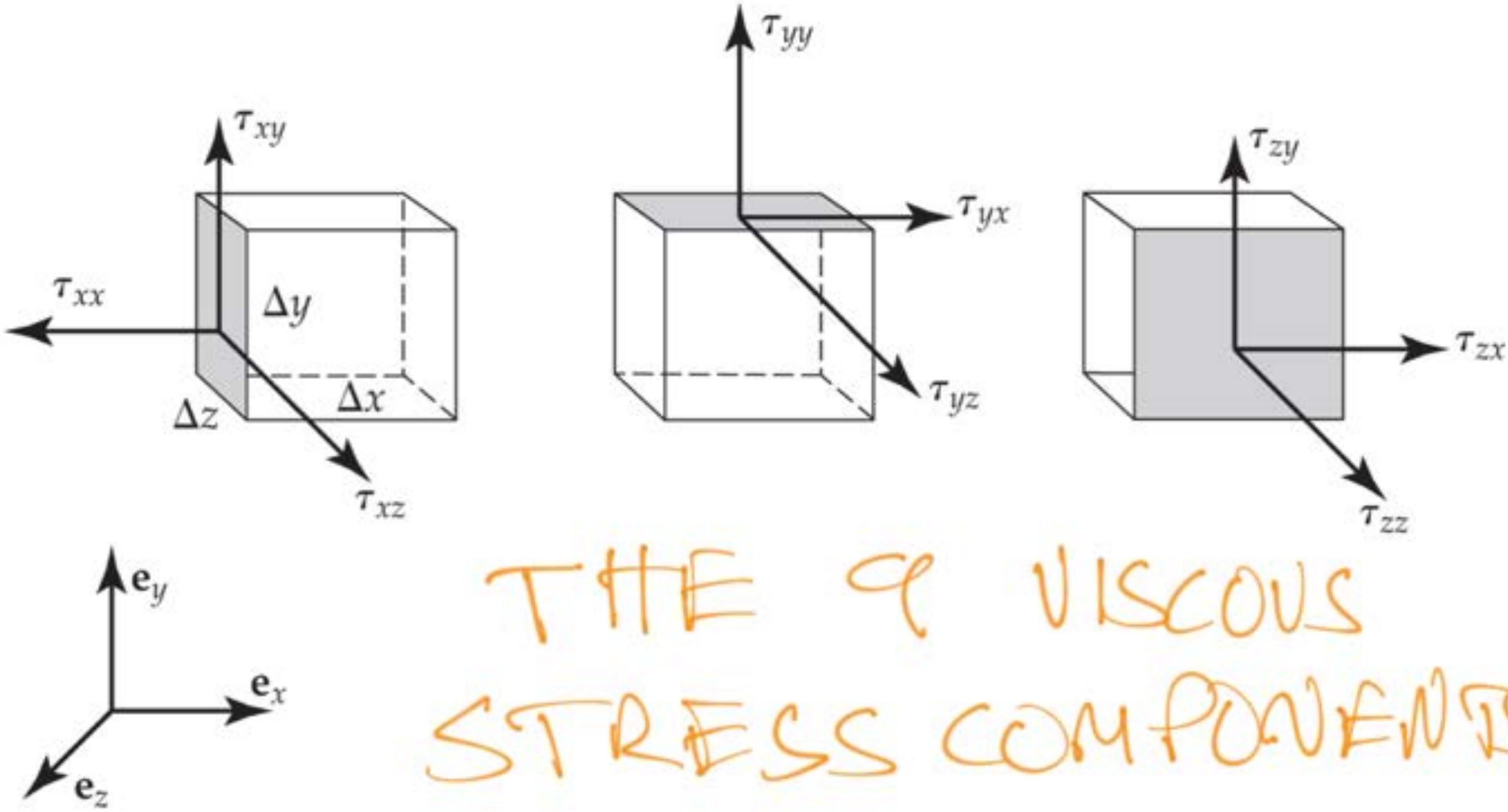
σ_{yx}
"SHEAR" (TANGENTIAL) COMPONENT

PEARSON
Transport Phenomena in Biological Systems, Second Edition
George A. Truskey, Fan Yuan, and David F. Katz

σ_{yx} ← DIRECTION

FACE

Figure 3.5 Schematic of components of the viscous stress tensor.



THE 9 VISCOUS STRESS COMPONENTS

$$\bar{\bar{\sigma}} = \bar{\bar{\tau}} - \bar{\bar{I}} p$$

COMPLETE EXPRESSION FOR SURFACE FORCES

$$\vec{\vec{\sigma}} = \vec{\vec{\tau}} - \vec{\vec{I}}\,p$$

PRESSURE

ALL SURFACE FORCES

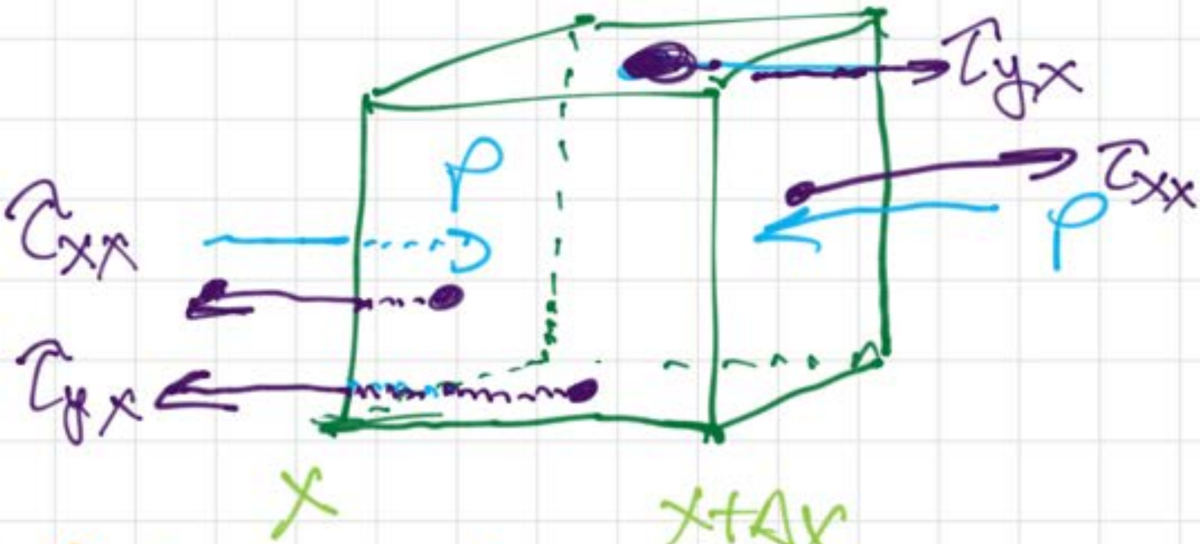
VISCOUS SURFACE FORCES

$$\vec{\vec{\sigma}} = \sigma_{ij} = \begin{pmatrix} \tau_{xx}-p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy}-p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz}-p \end{pmatrix}$$

TANGENTIAL COMPONENTS

NORMAL COMPONENTS

FORCE BALANCE FOR CUBE



X-DIRECTION:

$$(\tau_{xx} - p)\Big|_{x+\Delta x} \Delta y \Delta z - (\tau_{xx} - p)\Big|_{x} \Delta y \Delta z$$

$$\tau_{yx}\Big|_{y+\Delta y} \Delta z \Delta x - \tau_{yx}\Big|_{y} \Delta z \Delta x$$

$$\tau_{zx}\Big|_{z+\Delta z} \Delta x \Delta y - \tau_{zx}\Big|_{z} \Delta x \Delta y$$

SURFACE IN X-DIRECTION

$$F_{Sx} = -\left(P_{x+\Delta x} - P_x\right)\Delta y \Delta z +$$

$$\left(\tau_{xx}\Big|_{x+\Delta x} - \tau_{xx}\Big|_x\right)\Delta y \Delta z +$$

$$\left(\tau_{yx}\Big|_{y+\Delta y} - \tau_{yx}\Big|_y\right)\Delta z \Delta x +$$

$$\left(\tau_{zx}\Big|_{z+\Delta z} - \tau_{zx}\Big|_z\right)\Delta x \Delta y$$

F_{Sy}, F_{Sz}

PUTTING THIS ALL TOGETHER ...

$$dV\left(\rho\frac{\partial \vec{v}}{\partial t}+\rho\vec{v}\cdot\vec{\nabla}\vec{v}\right)=$$

$$dV\left(-\vec{\nabla}P+\vec{\nabla}\cdot\bar{\bar{\tau}}+\rho\vec{g}\right)$$

WHICH GIVES...

$$\rho\frac{\partial \vec{v}}{\partial t}+\rho\vec{v}\cdot\vec{\nabla}\vec{v}=-\vec{\nabla}P+\vec{\nabla}\cdot\bar{\bar{\tau}}+\rho\vec{g}$$

CAUCHY - MOMENTUM EQS

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

y component

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

"CAUCHY-MOMENTUM EQUATIONS"