CBE 30357
TEST 1 REVIEW
9/26/17

MAIN EDUCATIONAL GOAL:

LEARN TO SOLVE DIFFERENTIAL FLUID-FLOW PROBLEMS FOR STEADY- SINGLE DIRECTION FLOWS

THIS IS THE MATERIAL THAT WILL BE TESTED!!

YOUR PRIMARY "ANSWER" IS A VELOCITY PROFILE THAT RESULTS FROM INTEGRATING THE DIFFERENTIAL EQ.

$v_x(y) = \ldots$

$v_\theta(r) = \ldots$

$v_z(r) = \ldots$

YOU CAN THEN GET:

SHEAR STRESS: (FROM A DERIVATIVE)

TOTAL FLOW: (INTEGRATE VELOCITY)

PRESSURE GRADIENT: (REARRANGE EQ FOR TOTAL FLOW)

THE EQUATIONS :

MASS CONSERVATION

TABLE 3.1

The Conservation of Mass (Continuity Equation)

Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho \mathrm{v}_x}{\partial x} + \frac{\partial \rho \mathrm{v}_y}{\partial y} + \frac{\partial \rho \mathrm{v}_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r \mathrm{v}_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho \mathrm{v}_\theta)}{\partial \theta} + \frac{\partial(\rho \mathrm{v}_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 \mathrm{v}_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\rho \mathrm{v}_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial(\rho \mathrm{v}_\phi)}{\partial \phi}\right)$

FOR STRAIGHT, STEADY 1-D FLOWS, NOT MUCH INSIGHT IS GAINED.

WE SHOWED THAT THE ENERGY BALANCE COULD NOT PROVIDE

FLOWRATE <=> PRESSURE DROP

RELATION FOR PIPE FLOW

SO WE REALIZED THAT WE NEEDED

CONSERVATION OF MOMENTUM

$$\boxed{\bar{F} = m\bar{a}}$$

$$m\bar{a} = \Sigma \bar{F}$$

LEFT SIDE OF N.S. EQ

RIGHT SIDE OF N.S EQ'S

$$\frac{d(m\bar{v})}{dt} = \frac{d\bar{P}}{dt} = \Sigma \bar{F}$$

WE APPLY THIS "LAW" TO THE CASE OF AN ARBITRARY "CUBE"

Figure 3.1 Flow across a surface of constant x for a rectangular control volume.

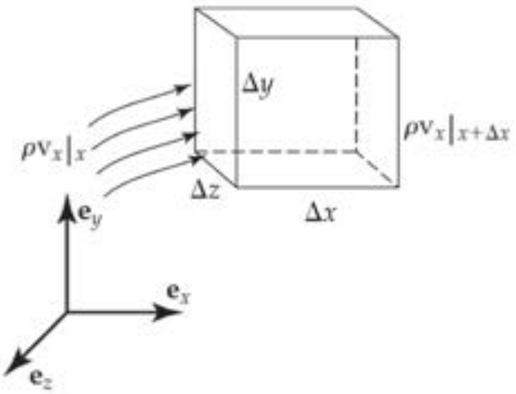
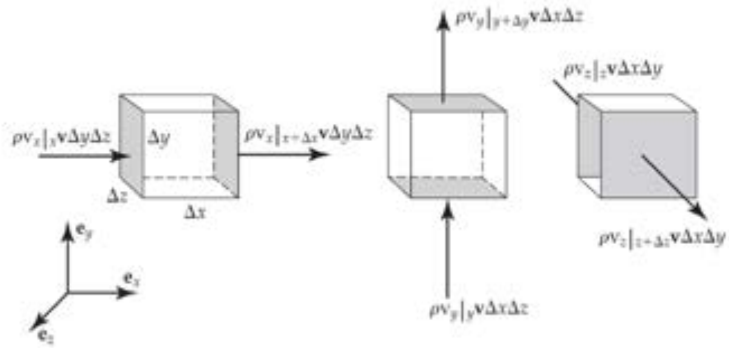
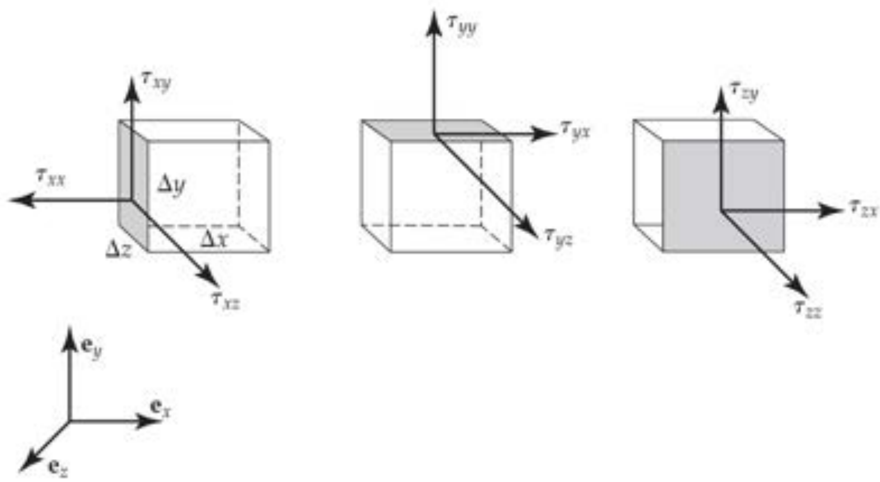


Figure 3.4 Momentum transport across surfaces of cubic control volume.



PEARSON *Transport Phenomena in Biolog...* George A. Truskey, Fan Yuan, ...

Figure 3.5 Schematic of components of the viscous stress tensor.



PEARSON *Transport Phenomena in Biological Systems*, Second Edition George A. Truskey, Fan Yuan, and David F. Katz

THEN SOME APPROPRIATE PRESTIDIGITATION...

able 3.4

TABLE 3.4 NEWTONIAN FLUID

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

PRESSURE

GRAVITY

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\underbrace{\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]}_{\text{VISCOUS STRESS}} + \rho g_x$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

$$\frac{d\bar{P}}{dt} = \sum \bar{F}$$

FOR STRAIGHT LINE FLOWS LEFT SIDE TERMS ARE IDENTICALLY ZERO

YOU SELECT APPROPRIATE NON-ZERO RIGHT SIDE TERMS TO SOLVE

HOW TO CHOOSE?

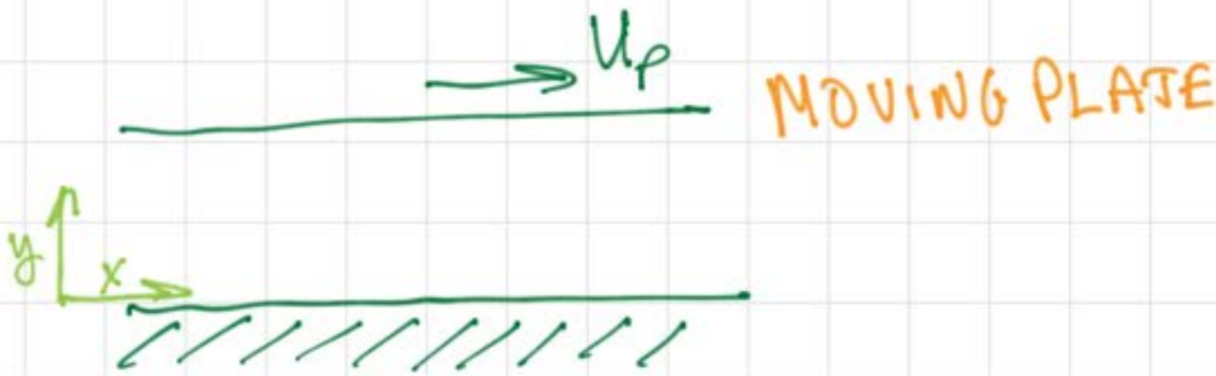
FLOWS CAN BE CAUSED BY:

1.) MOVING SURFACES

2.) PRESSURE GRADIENTS

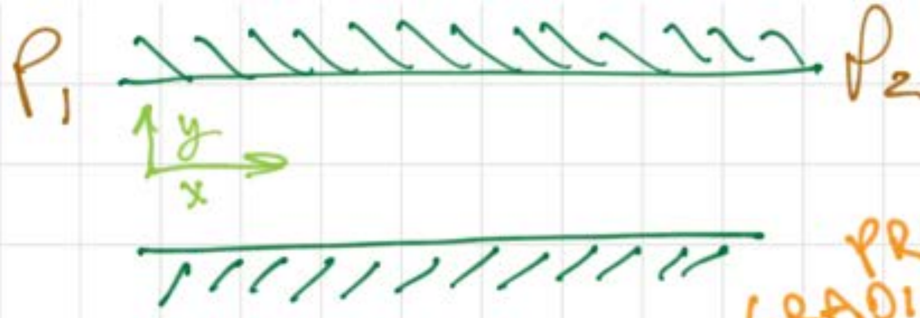
3.) GRAVITY

1)



NO PRESSURE GRADIENT

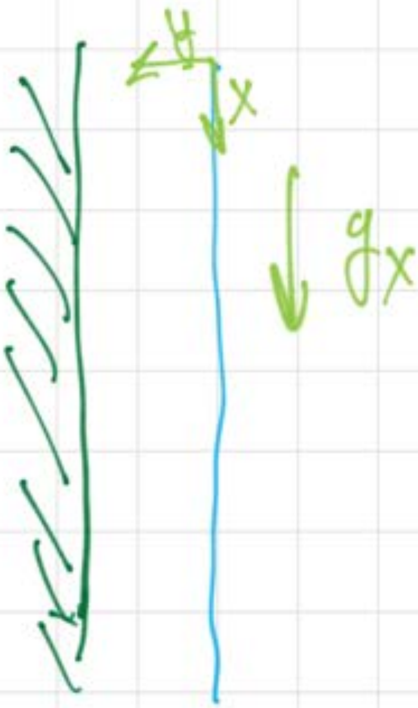
$$0 = \mu \frac{\partial^2 V_x}{\partial y^2}$$

2) 

PRESSURE GRADIENT CAUSES FLOW

$P_1 > P_2$

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

3) 

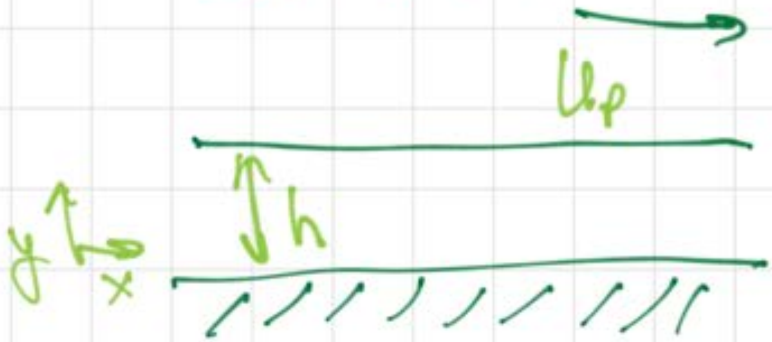
$$0 = \mu \frac{\partial^2 v_x}{\partial y^2} + \rho g_x$$

GRAVITY CAUSES FLOW

NEXT WE NEED BOUNDARY CONDITIONS

BOUNDARY CONDITIONS

1) FLUIDS "STICK" TO SOLID SURFACES "NO-SLIP"



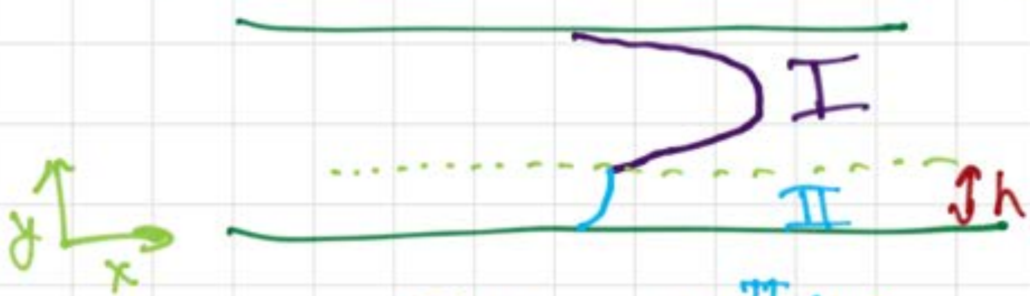
$V_x(0) = 0$

$V_x(h) = u_p$

2) AT A FLUID-FLUID INTERFACE

VELOCITIES FOR EACH FLUID ARE THE SAME

FLUID STICKS TO OTHER FLUID



$V_x^{I}(h) = V_x^{II}(h)$

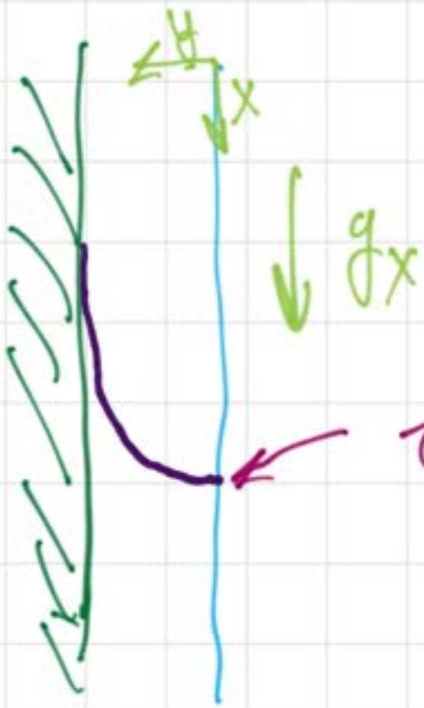
3) SHEAR STRESS MATCHES AT FLUID - FLUID INTERFACE

$\tau_{yx}(y)$ $\boxed{\text{DOES NOT DEPEND ON FLUID}}$!!

$$\therefore \quad \tau_{yx}^{I}(h) = \tau_{yx}^{II}(h)$$

$$\underbrace{\mu^{I} \left.\frac{\partial v_x^{I}}{\partial y}\right|_h = \mu^{II} \left.\frac{\partial v_x^{II}}{\partial y}\right|_h}_{\text{IF NEWTONIAN}}$$

4) $\tau_{yx} \simeq 0$ AT AIR-LIQUID INTERFACE



$\tau_{yx} = 0$ $\quad \frac{\partial v_x}{\partial y} = 0$

SOME FLOWS

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

PRESSURE

GRAVITY

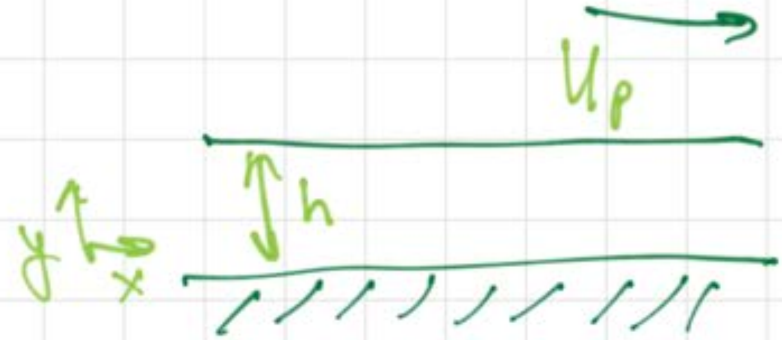
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \underbrace{\mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]}_{\text{VISCOUS STRESS}} + \rho g_x$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$



$$0 = \mu\frac{\partial^2 v_x}{\partial y^2}$$

$$v_x(0) = 0$$
$$v_x(h) = U_p$$

$$\frac{d}{dy}\left(\frac{dv_x}{dy}\right) = 0$$

$$d\left(\frac{dv_x}{dy}\right) = 0$$

$$\int d\,\frac{dv_x}{dy} = \int 0$$

$$\frac{dv_x}{dy} + C_1 = 0$$

able 3.4

TABLE 3.4 NEWTONIAN FLUID

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

PRESSURE GRAVITY

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \underbrace{\mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]}_{\text{VISCOUS STRESS}} + \rho g_x$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

$$\frac{dV_x}{dy} = C_1$$

$$d(V_x) = C_1 \, dy$$

$$\int d(V_x) = \int C_1 \, dy$$

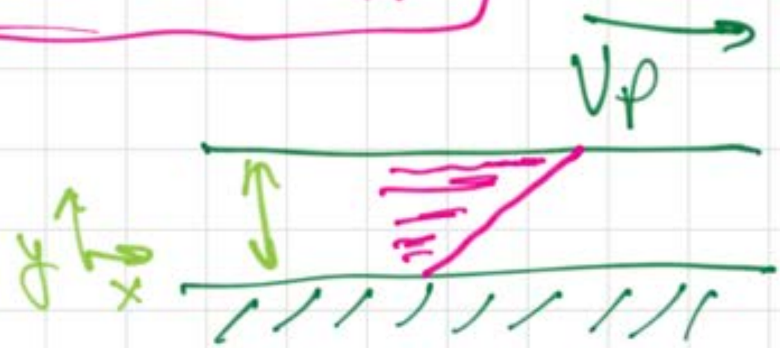
$$V_x = C_1 y + C_2$$

LINEAR PROFILE

$$V_x(0) = 0 \quad \therefore C_2 = 0$$

$$V_x(h) = U_p \quad \therefore C_1 = \frac{U_p}{h}$$

$$\boxed{V_x = U_p \frac{y}{h}}$$



VOLUMETRIC FLOW

W = WIDTH

$$Q = W\int_0^h v_x(y)\,dy = W\int_0^h U_P\left(\frac{y}{h}\right)dy$$

$$Q = W U_P \left.\frac{y^2}{2h}\right|_0^h = W U_P \frac{h}{2}$$

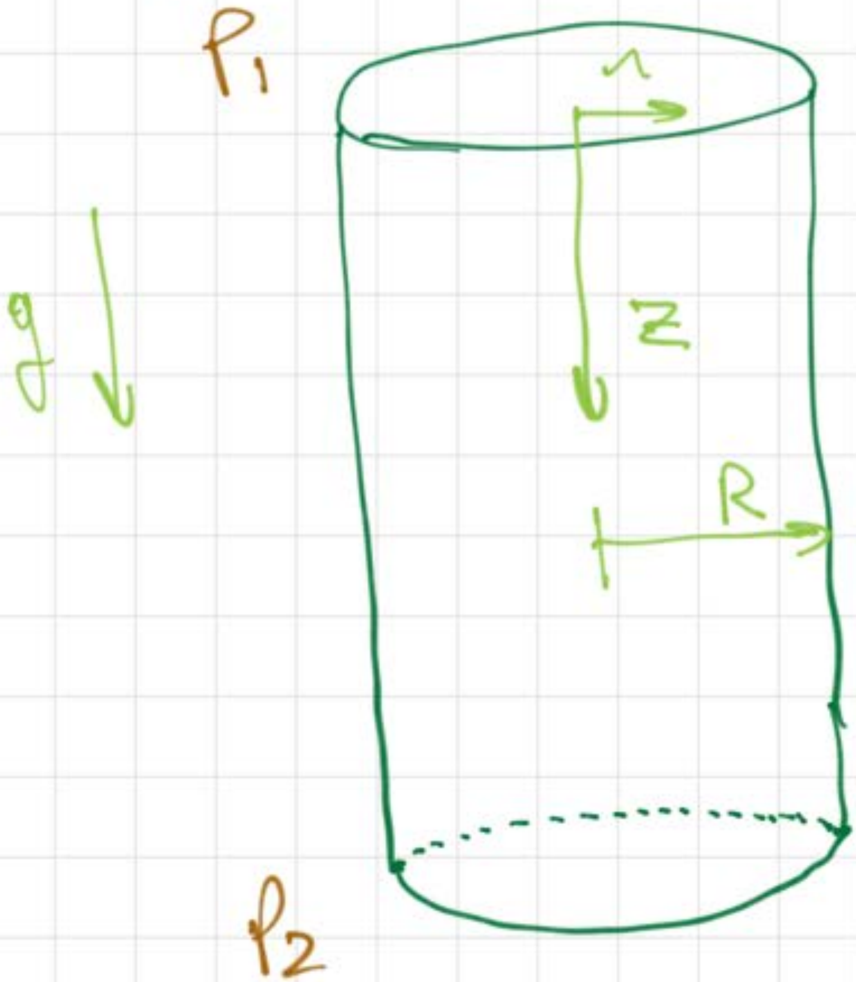
AVERAGE VELOCITY

$$\langle v \rangle = \frac{Q}{A} = \frac{W U_P h/2}{W h}$$

$$\langle v \rangle = \frac{U_P}{2}$$

FLOW IN A CIRCULAR TUBE
PRESSURE + GRAVITY

$P_1 > P_2$



BC'S $v_z(R)=0$ NO SLIP

$v_z'(0)=0$ SYMMETRY

Cylindrical coordinates

r direction r

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r}\frac{\partial v_r}{\partial r} + \frac{\cancel{v_\theta}}{r}\cancel{\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + v_z\frac{\partial v_r}{\cancel{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial r}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r\cancel{v_r})}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right] + \cancel{\rho g_r}$$

θ direction θ

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r}\cancel{\frac{\partial v_\theta}{\partial r}} + \frac{\cancel{v_\theta}}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}}\right) = -\frac{1}{r}\frac{\partial p}{\cancel{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r\cancel{v_\theta})}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancel{\rho g_\theta}$$

z direction z

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \frac{\cancel{v_\theta}}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

$$0 = -\frac{\partial P}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\partial P}{\partial z} - \rho g_z = \text{CONST} \equiv G$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{G}{\mu}$$

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{G}{\mu} r$$

$$d\left(r\frac{dv_z}{dr}\right) = \frac{G}{\mu} r\,dr$$

$$\int d\left(r\frac{dv_z}{dr}\right) = \frac{G}{\mu}\int r\,dr$$

$$r\frac{dv_z}{dr} = \frac{G}{\mu}\frac{r^2}{2} + C_1$$

$$\frac{dv_z}{dr} = \frac{G}{\mu}\frac{r}{2} + \frac{C_1}{r}$$

$$\int dv_z = \int\left(\frac{G}{\mu}\frac{r}{2} + \frac{C_1}{r}\right)dr$$

$$v_z = \frac{G}{\mu}\frac{r^2}{4} + C_1 \ln r + C_2$$

FIT B.C'S

$$V_z'(r=0) = 0 \quad \therefore C_1 = 0$$

$$V_z(R) = 0 = \frac{G}{\mu}\frac{R^2}{4} + C_2$$

$$\therefore C_2 = -\frac{G}{\mu}\frac{R^2}{4}$$

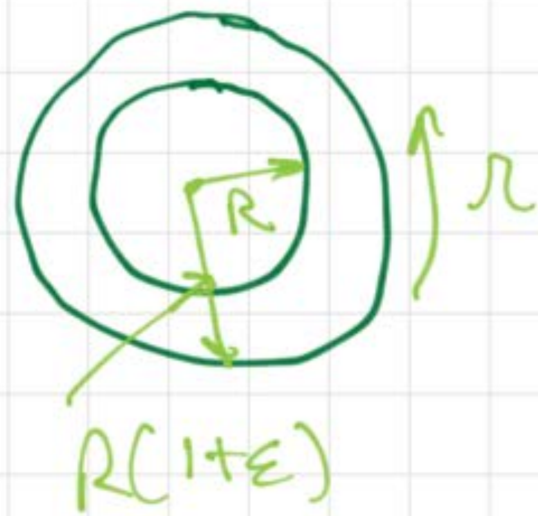
$$\therefore V_z(r) = -\frac{G}{4\mu}R^2\left(1 - \frac{r^2}{R^2}\right)$$

RECALL

$$G = \frac{dP}{dz} - \rho g_z$$

SOLUTION WORKS FOR ANY COMBINATION OF ΔP + g_z

HOW ABOUT A ROTATING FLOW ...



$$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

$$v_\theta(R) = 0$$

$$v_\theta(R(1+\varepsilon)) = R(1+\varepsilon)\Omega$$

$$v_\theta = \frac{c_1}{2} r + \frac{c_2}{r}$$

ETC.

IF FLOW IS NON NEWTONIAN

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

y component

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]$$

z component

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

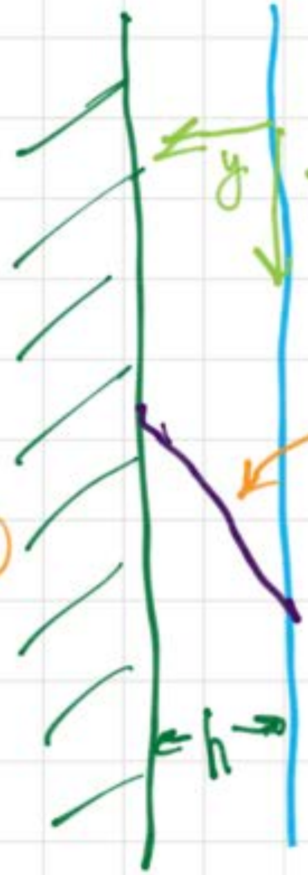
$$0 = \frac{\partial \tau_{yx}}{\partial y} + \rho g_x$$

$$\frac{d\tau_{yx}}{dy} = -\rho g_x$$

$$\int d(\tau_{yx}) = -\rho g_x \int dy$$

$$\tau_{yx} = -\rho g_x y + C_1 \qquad \tau_{yx}(0) = 0$$

$$\boxed{\tau_{yx} = -\rho g_x y} \qquad \therefore C_1 = 0$$



NO FLOW

$$0 = \frac{\partial \tau_{yx}}{\partial y} + \rho g_x$$

$$\tau_{yx} = -\rho g_x y$$

$$\tau_{yx}(h) = -\rho g_x h$$

IF $|\tau_{yx}(h)| < \tau_0$

$$|\tau_{yx}(h)| > \tau_0$$

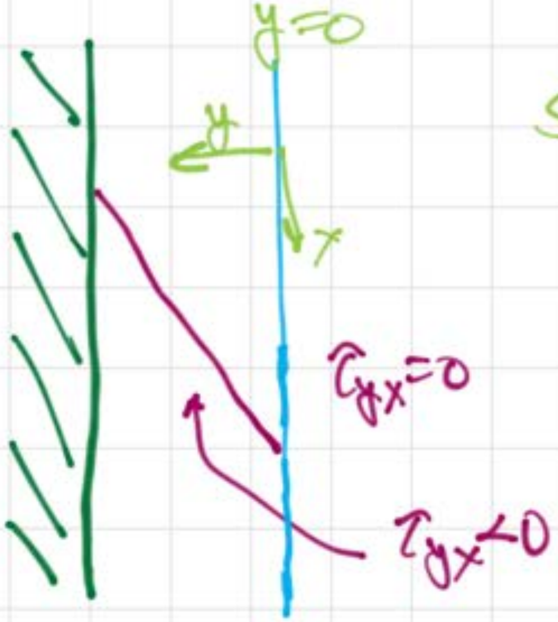
$$\tau_{yx}(y) = \tau_0$$

FLAT

$$\tau_{yx} = -\rho g_x y$$

FOR A BINGHAM PLASTIC:

$$\tau_{yx} = -\tau_0 + \mu \frac{\partial v_x}{\partial y}$$



SO y IS POSITIVE INWARD

SUBS CONSTITUTIVE RELATION INTO INTEGRATED MOMENTUM EQ.

$$-\tau_0 + \mu \frac{\partial v_x}{\partial y} = -\rho g_x y$$

$$\mu \frac{\partial v_x}{\partial y} = -\rho g_x y + \tau_0$$

$$\int dv_x = \int \left(\frac{-\rho g_x y}{\mu} + \tau_0\right) dy$$

$$v_x = \frac{-\rho g_x}{\mu}\frac{y^2}{2} + \frac{\tau_0}{\mu} y + C_1$$

$$v_x(h) = 0$$

$$\therefore \quad C_1 = \frac{\rho g_x}{\mu}\frac{h^2}{2} - \frac{\tau_0}{\mu} h$$

$$\therefore \quad v_x = \frac{\rho g_x}{2\mu}\left(h^2 - y^2\right) + \frac{\tau_0}{\mu}(y - h)$$

VALID IN $|\tau_{yx}| > |\tau_0|$

FOR $\tau_{yx} \leq \tau_0$

$$v_x = v_{MAX}$$

HOW DO WE FIND $y_0 \Rightarrow (\tau_{yx} = \tau_0)$?

FIND $\frac{dv_x}{dy} = 0$

$$\frac{dv_x}{dy} = -\frac{\rho g_x}{\mu} y + \frac{\tau_0}{\mu} = 0$$

$$y_0 \equiv y\Big|_{\tau=\tau_0} = \frac{\tau_0}{\rho g_x}$$

SO FOR $y < y_0$

$$v_x = \frac{\rho g_x}{2\mu}\left(h^2 - y^2\right) + \frac{\tau_0}{\mu}(y - h)$$

$$v_{x\,MAX} = \frac{\rho g_x}{2\mu}\left(h^2 - \left(\frac{\tau_0}{\rho g_x}\right)^2\right) + \frac{\tau_0}{\mu}\left(\frac{\tau_0}{\rho g_x} - h\right)$$

$$v_{x\,MAX} = \frac{\left(\tau_0 - \rho g_x h\right)^2}{2 g_x \mu \rho}$$

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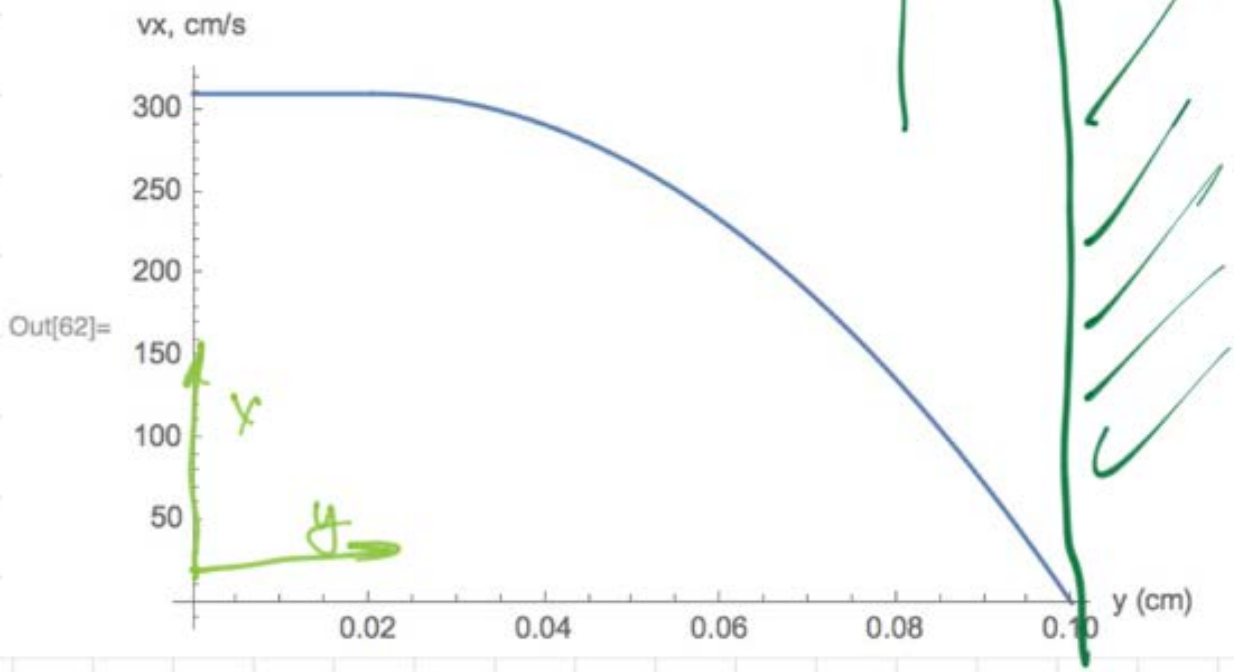

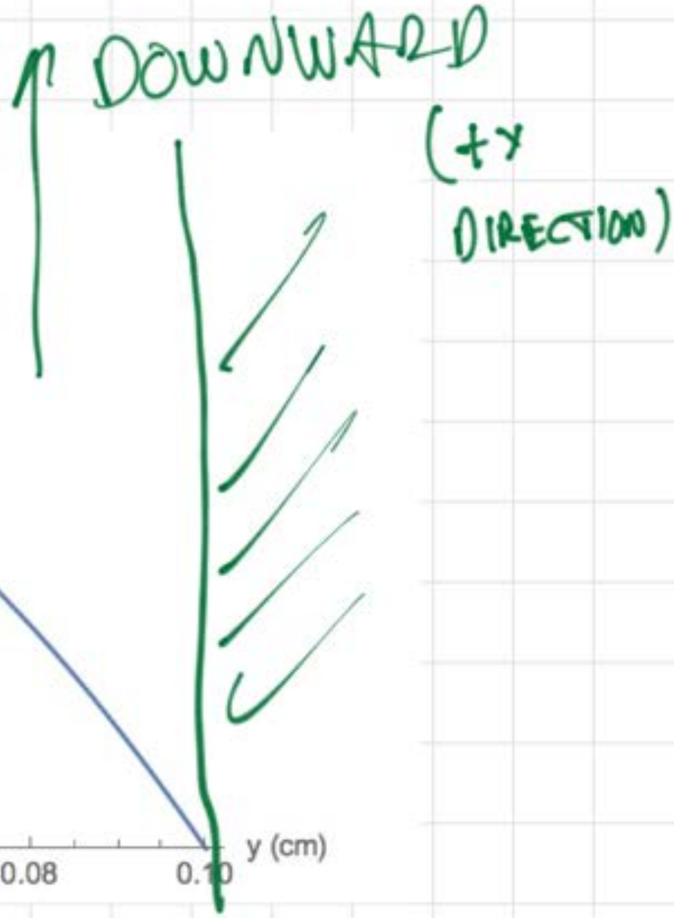


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Plot[veex /. {gx → 980, ρ → 1, h → .1, τ0 → 20, μ → .01}, {y, 20 / 980, .1},
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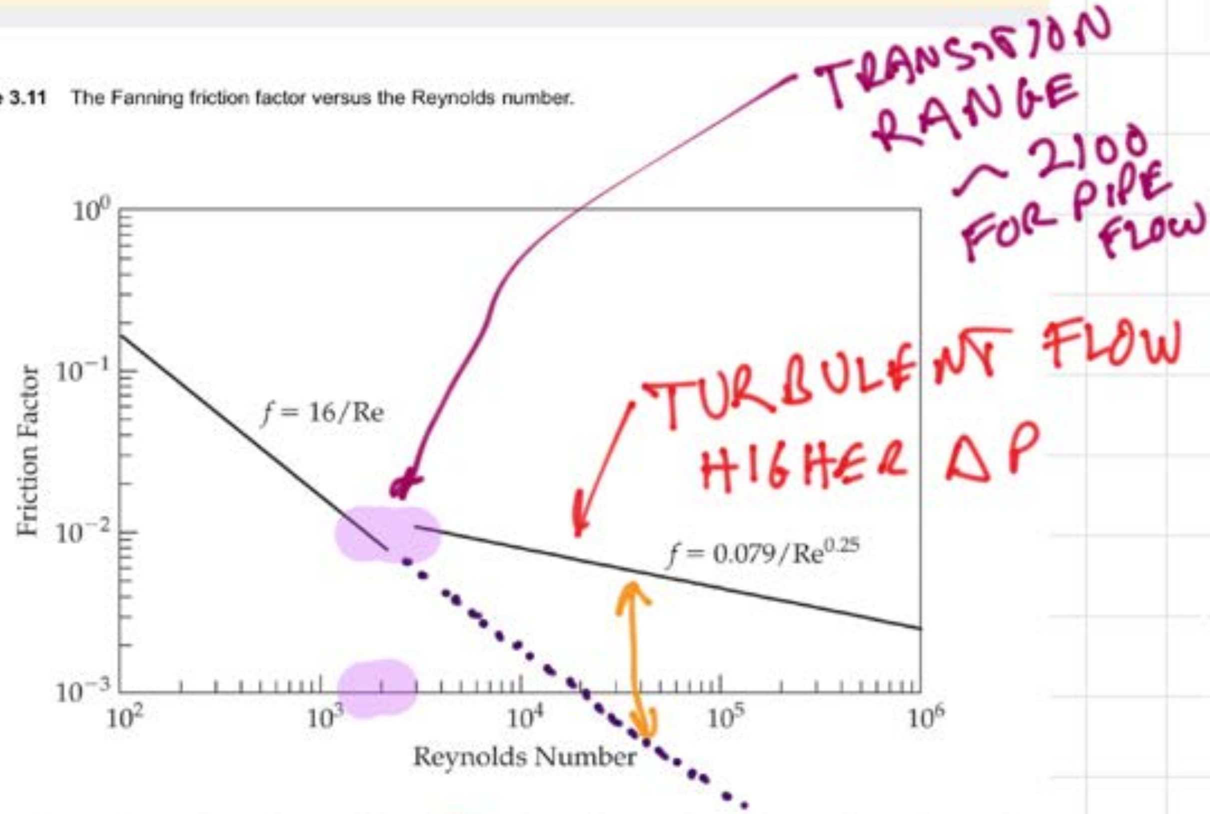
TO GET FLOWRATE INTEGRATE
2 PIECES

$$Q = W \int_0^{\tau_0/\rho g_x} V_{MAX}\, dy +$$

$$W \int_{\tau_0/\rho g_x}^{h} \left(\frac{\rho g_x}{2\mu}\left(h^2 - y^2\right) + \frac{\tau_0}{\mu}(y - h) \right) dy$$

LAMINAR + TURBULENT FLOW

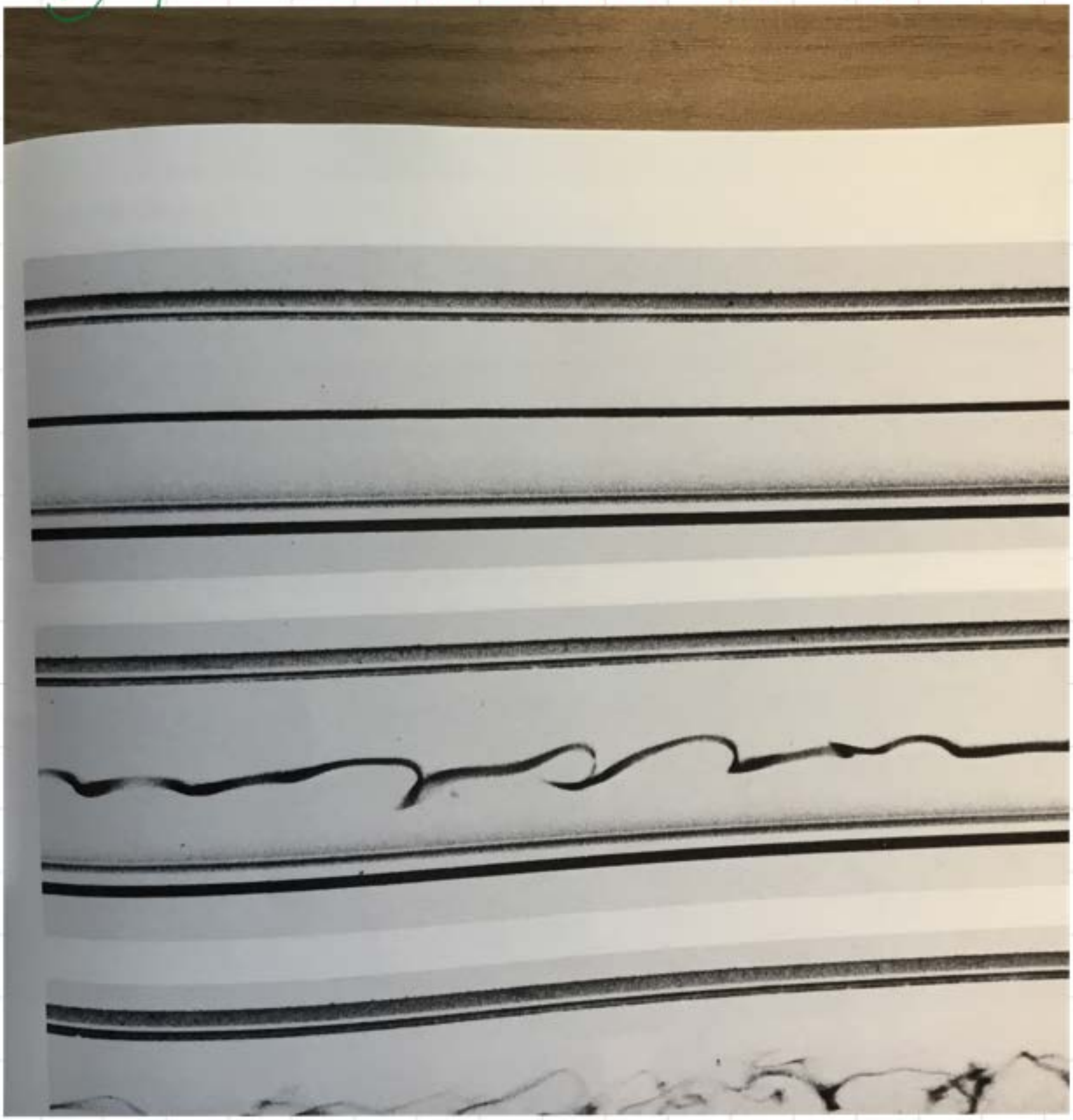
Figure 3.11 The Fanning friction factor versus the Reynolds number.



$$f \equiv \frac{\Delta P\, D}{2 L \rho v^2}$$

Turbulent velocity fluctuations “mix” slow moving and fast moving fluid and in doing so transfer the stress from the wall, that is opposing flow, into the middle of the pipe. Result: high pressure drop for a given flow than if the flow were laminar.

DYE STREAM IN PIPE FLOW



103. **Repetition of Reynolds' dye experiment.** Osborne Reynolds' celebrated 1883 investigation of stability of flow in a tube was documented by sketches rather than photography. However the original apparatus has survived at the University of Manchester. Using it a century later, N. H. Johannesen and C. Lowe have taken this sequence of photographs. In laminar flow a filament of colored water introduced at a bell-shaped entry extends undisturbed the whole length of the glass tube. Transition is seen in the second of the photographs as the speed is increased; and the last two photographs show fully turbulent flow. Modern traffic in the streets of Manchester made the critical Reynolds number lower than the value 13,000 found by Reynolds.

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AS Re ↑ MORE "MIXING"

STRONGER DISTURBANCES