CBE 30357
9/19/17

TOPICS

1) REVIEW: BOUNDARY CONDITIONS & TWO-LAYER FLOWS

2) SIMPLE ROTATING FLOW
$\frac{d\bar{P}}{dt} \neq 0$!!

3) CATHETER PROBLEM:
LIMIT $x \Rightarrow 0$

4) LAMINAR-TURBULENT TRANSITION

5) NON-NEWTONIAN LIQUIDS

Boundary conditions:

1. Fluid sticks to solid surfaces $V_x(0) = 0$
2. Fluid sticks to another immiscible fluid $V_x^{I}(h) = V_x^{II}(h)$
3. The shear stress is continuous across the fluid-fluid interface

$$\tau_{yx}^{I}(h) = \tau_{yx}^{II}(h)$$

$$\mu^{I} \left.\frac{\partial V_x^{I}}{\partial y}\right|_h = \mu^{II} \frac{\partial V_x^{II}}{\partial y}$$

4. For liquid flowing next to quiescent air, the stress can be considered approximately 0.

$$\tau_{yx}(h) \approx 0$$

$$\frac{\partial V_x}{\partial y}(h) \approx 0$$

"FALLING FILM"

SUPPOSE 2 IMMISCIBLE OR SEPARATE LAYERS ARE FLOWING TOGETHER

2 SEPARATE EQUATIONS

$$0 = -\frac{\partial P}{\partial x} + \mu^{I} \frac{\partial^2 v_x^{I}}{\partial y^2} \qquad v_x^{I}(h) = v_x^{II}(h)$$

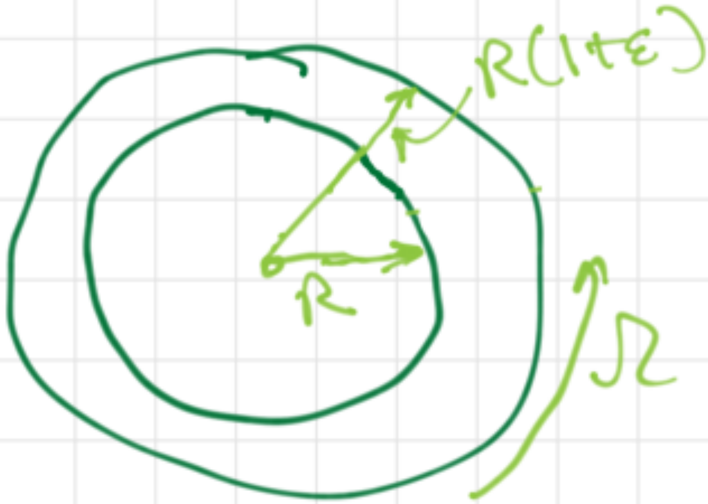
$$0 = -\frac{\partial P}{\partial x} + \mu^{II} \frac{\partial^2 v_x^{II}}{\partial y^2} \qquad \mu^{I} \left.\frac{\partial v_x^{I}}{\partial y}\right|_h = \mu^{II} \frac{\partial v_x^{II}}{\partial y}$$

PRESSURE GRADIENTS ARE SAME (OR WOULD BE 0 IF NO ΔP IS IMPOSED)

4 B.C'S. 2 SECOND ORDER ODES.

SOLVE

ROTATING FLOW



CONCENTRIC CYLINDERS

θ DIRECTION MOTION, $V_\theta \neq 0$

BUT WE HAVE θ - SYMMETRY

$$\frac{\partial [\,?\,]}{\partial \theta} = 0$$

Cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \quad (3.3.27a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \quad (3.3.27b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad (3.3.27c)$$

$$0 = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rV_\theta)}{\partial r}\right)$$

$$V_\theta(R) = 0$$

$$V_\theta(R(1+\epsilon)) = R(1+\epsilon)\Omega$$

SOLVE

$$\frac{1}{r}\frac{d}{dr}(rV_\theta) = C_1$$

$$\frac{d}{dr}(rV_\theta) = C_1 r$$

CORRECTED AFTER LECTURE

$$\boxed{V_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}}$$

NO r=0 SO BOTH TERMS ARE OK.

FIT B.C.'S

$$0 = \frac{C_1 R}{2} + \frac{C_2}{R}$$

$V_\theta(R) = 0$

$$C_2 = -\frac{C_1 R^2}{2}$$

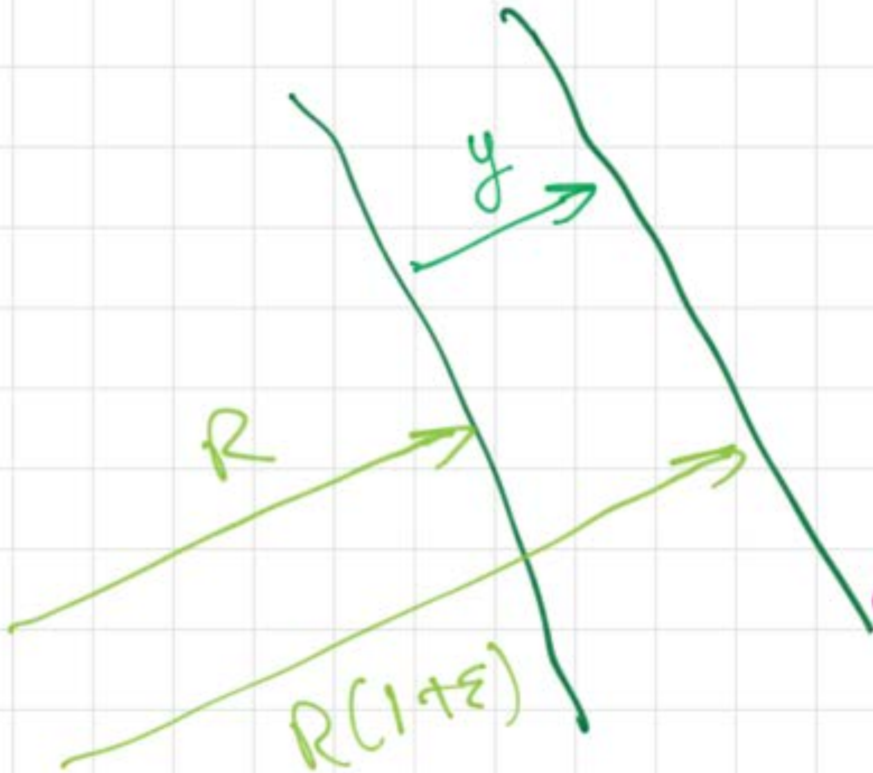
$$R(1+\epsilon)\Omega = C_1 \frac{R(1+\epsilon)}{2} - \frac{C_1 R^2}{2R(1+\epsilon)}$$

$$1 = \frac{C_1}{2\Omega}\left(1 - \frac{1}{(1+\epsilon)^2}\right)$$

$V_\theta(R(1+\epsilon)) = R(1+\epsilon)\Omega$

$$C_1 = \frac{2\Omega}{\left(1 - \frac{1}{(1+\epsilon)^2}\right)}$$

$$V_\theta(r) = \frac{\Omega}{\left(1 - \frac{1}{(1+\varepsilon)^2}\right)}\left(r - \frac{R^2}{r}\right)$$



$0 \le y \le 1$

$r = R + \varepsilon R y$

or: $y = \dfrac{r - R}{\varepsilon R}$

SERIES EXPANSION IN ε

$$V_\theta(y) = R\Omega y + \frac{\varepsilon}{2}\left(\frac{3Ry\Omega}{2} - \frac{1}{2}Ry^2\Omega\right)$$

AS $\varepsilon \to 0$ LOOKS LIKE PARALLEL PLATE

USING WOLFRAM ALPHA TO GET SERIES EXPANSION

7:37 AM 81%

WolframAlpha

substitute r=R+R*eps*y into (r-R)*(r+R)*(1+eps)^2*omega/r/eps/(2+eps)

Input interpretation

$$(r-R)(r+R)(1+\epsilon)^2 \times \frac{\frac{\frac{\omega}{r}}{\epsilon}}{2+\epsilon} \text{ where } r = R + R\, y\, \epsilon$$

Result

$$\frac{R\, y\, \omega (\epsilon+1)^2 (R\, y\, \epsilon + 2R)}{(\epsilon+2)(R\, y\, \epsilon + R)}$$

Step-by-step solution

7:48 AM 79

WolframAlpha

taylor series R*y*omega*(eps+1)^2*(y*eps+2)/(eps+2)/(y*eps+1) in eps

Input interpretation

series	$R\, y\, \omega (\epsilon+1)^2 \times \frac{\frac{y\epsilon+2}{\epsilon+2}}{y\,\epsilon+1}$	point	$\epsilon = 0$

Series expansion at ϵ=0

$$R\, y\, \omega - \frac{1}{2}\epsilon (R (y-3) y\, \omega) + \frac{1}{4} R\, y \left(2y^2 - 3y + 1\right) \omega \epsilon^2 - \frac{1}{8}\epsilon^3 \left(R\, y \left(4y^3 - 6y^2 + y + 1\right)\omega\right) + \frac{1}{16} R\, y \left(8y^4 - 12y^3 + 2y^2 + y + 1\right)\omega \epsilon^4 - \frac{1}{32}\epsilon^5 \left(R\, y \left(16y^5 - 24y^4 + 4y^3 + 2y^2 + y + 1\right)\omega\right) + O(\epsilon^6)$$

(Taylor series)

More terms

8:00 AM 76%

WolframAlpha

dsolve(mu/r*(v'(r)+r*v''(r))-G=0,v(R)=0,v(lambda*R)=0)

Input

$$\left\{\frac{\mu}{r}\left(v'(r)+r\,v''(r)\right)-G=0,\; v(R)=0,\; v(\lambda R)=0\right\}$$

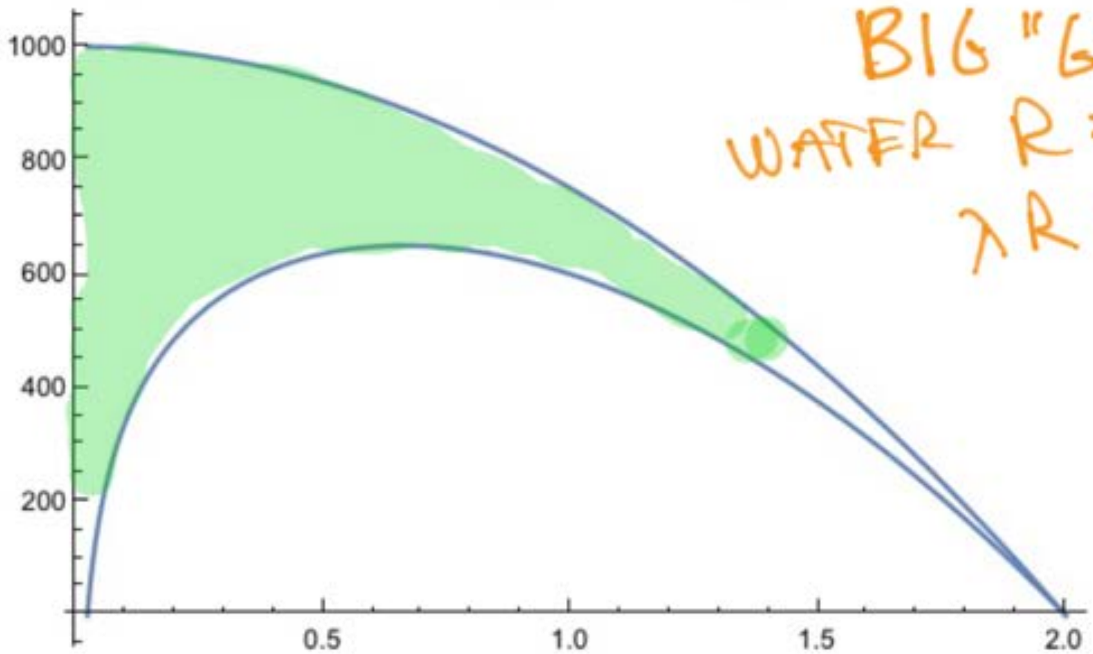
Differential equation solution

$$v(r)=\frac{G\left(\log(R)\left(r^2-\lambda^2 R^2\right)+\left(R^2-r^2\right)\log(\lambda R)+\left(\lambda^2-1\right)R^2\log(r)\right)}{4\,\mu\,(\log(R)-\log(\lambda R))}$$

Step-by-step solution

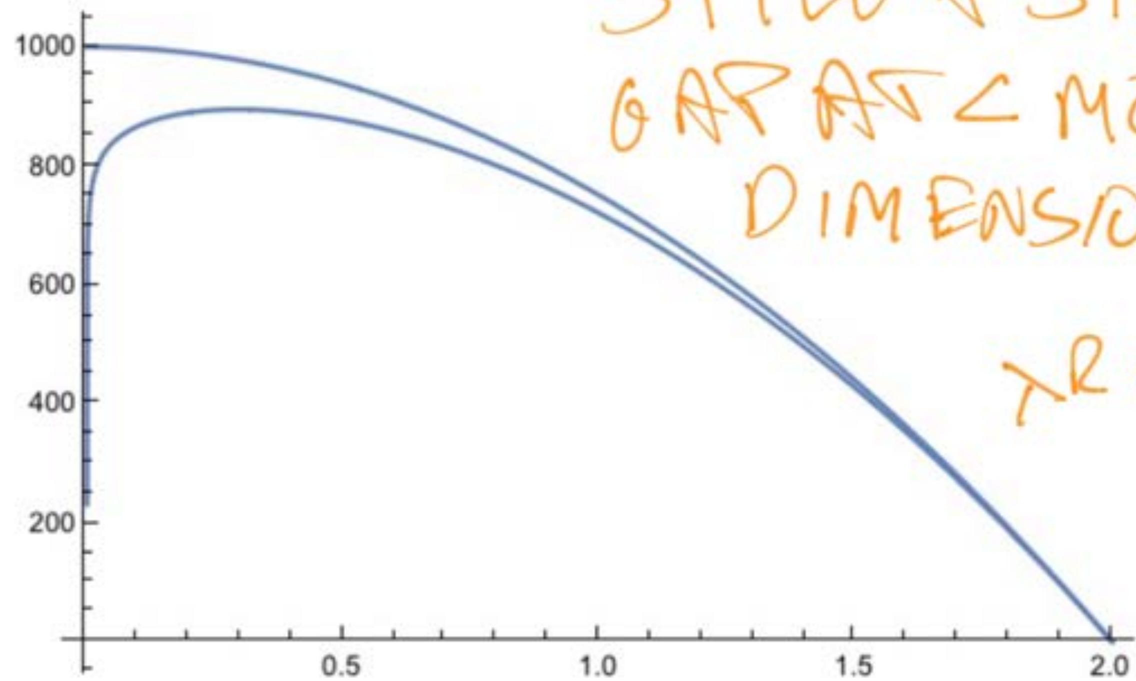
Related Wolfram|Alpha queries

```
Plot[{veez, vezee} /. {dpdz → -10, μ → .01, R → 2, λ → .01}, {r, .02, 2}]
```



BIG "GAP" FOR
WATER $R = 2\,cm$
$\lambda R = .02\,cm$

```
Plot[{veez, vezee} /. {dpdz → -10, μ → .01, R → 2, λ → .0000000001},
 {r, .0000000002, 2}]
```



STILL A SIGNIFICANT GAP AT < MOLECULAR DIMENSION !!

$\lambda R = 2 \times 10^{-10}$ cm

LIMIT OF OPEN CHANNEL EXISTS, BUT NOT RELEVANT PHYSICALLY

In[36]:= `Series[veez, {λ, 0, 2}]`

Out[36]=
$$\frac{1}{4\,\mu\,\mathrm{Log}[\lambda]}\left(\mathrm{dpdz}\,R^2\,\mathrm{Log}[r]-\mathrm{dpdz}\,R^2\,\mathrm{Log}[R]+\mathrm{dpdz}\,r^2\,\mathrm{Log}[\lambda]-\mathrm{dpdz}\,R^2\,\mathrm{Log}[\lambda]\right)+\frac{\left(-\mathrm{dpdz}\,R^2\,\mathrm{Log}[r]+\mathrm{dpdz}\,R^2\,\mathrm{Log}[R]\right)\lambda^2}{4\,\mu\,\mathrm{Log}[\lambda]}+O[\lambda]^3$$

In[37]:= `Apart[Normal[%]]`

Out[37]=
$$-\frac{\mathrm{dpdz}\left(-r^2+R^2\right)}{4\,\mu}-\frac{1}{4\,\mu\,\mathrm{Log}[\lambda]}\left(\mathrm{dpdz}\,R^2\,\mathrm{Log}[r]-\mathrm{dpdz}\,R^2\,\lambda^2\,\mathrm{Log}[r]-\mathrm{dpdz}\,R^2\,\mathrm{Log}[R]+\mathrm{dpdz}\,R^2\,\lambda^2\,\mathrm{Log}[R]\right)$$

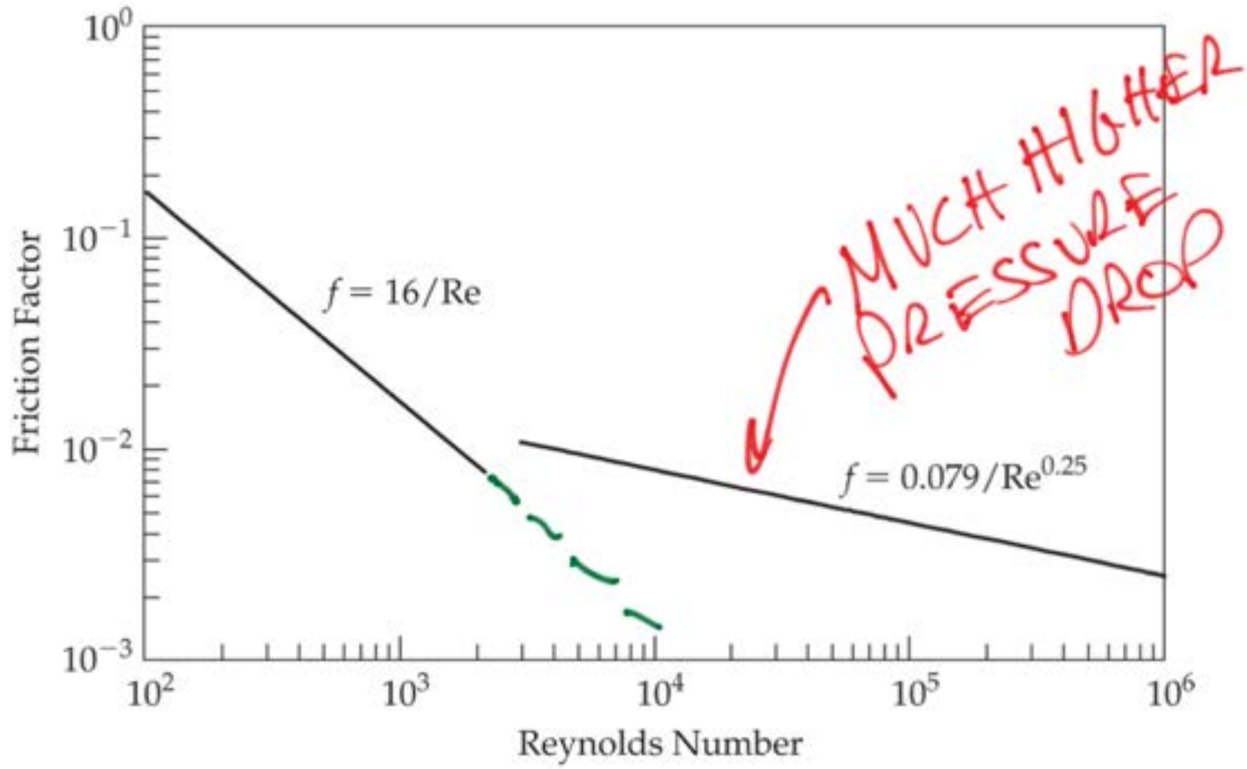
TRANSITION TO TURBULENT FLOW.

New Slide Layout

to newest file format.

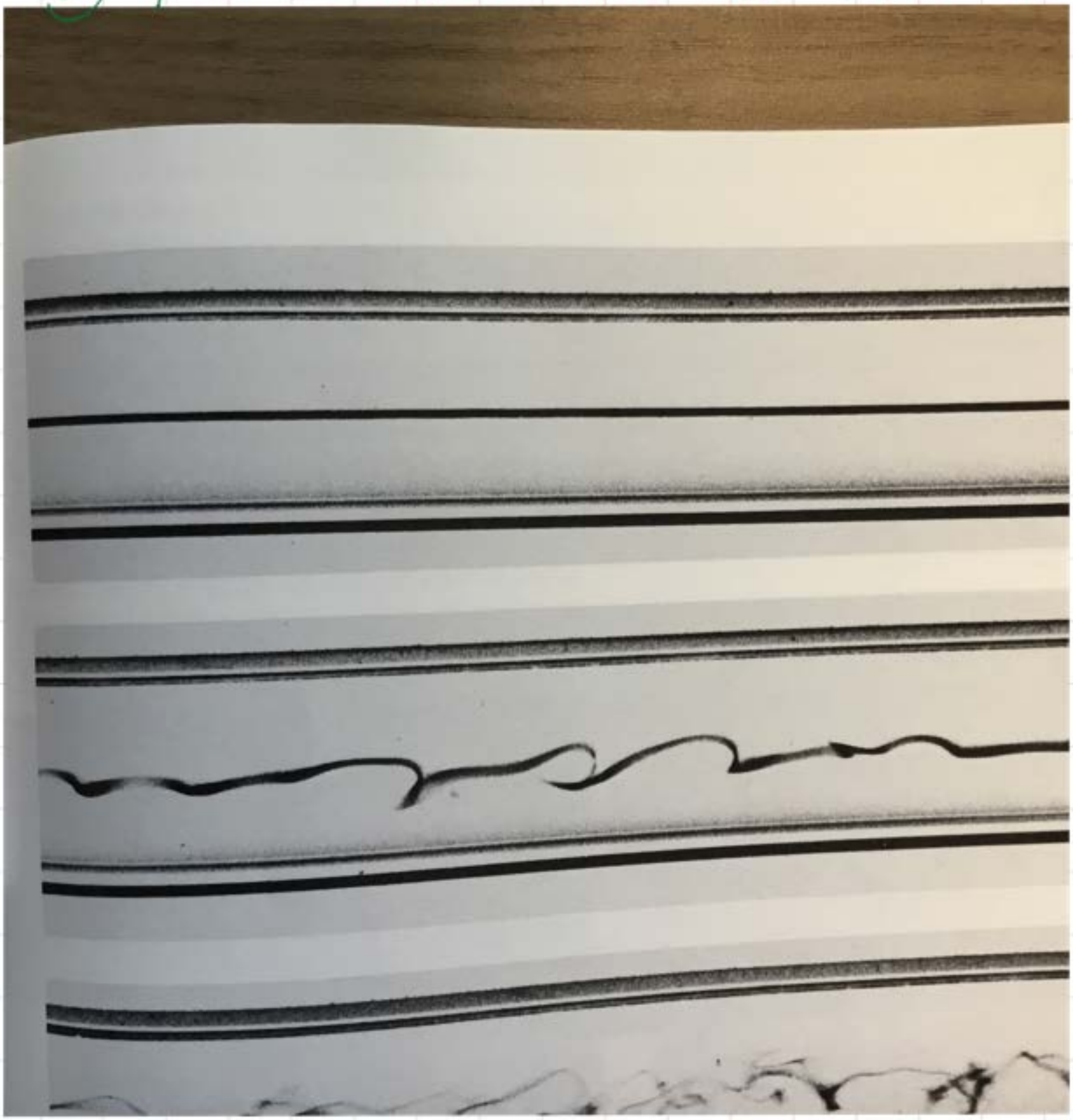
$$Re \equiv \frac{D v \rho}{\mu}$$

Figure 3.11 The Fanning friction factor versus the Reynolds number.



$$f \equiv \frac{\Delta P \, D}{2 L \rho v^2}$$

DYE STREAM IN PIPE FLOW



103. Repetition of Reynolds' dye experiment. Osborne Reynolds' celebrated 1883 investigation of stability of flow in a tube was documented by sketches rather than photography. However the original apparatus has survived at the University of Manchester. Using it a century later, N. H. Johannesen and C. Lowe have taken this sequence of photographs. In laminar flow a filament of colored water introduced at a bell-shaped entry extends undisturbed the whole length of the glass tube. Transition is seen in the second of the photographs as the speed is increased; and the last two photographs show fully turbulent flow. Modern traffic in the streets of Manchester made the critical Reynolds number lower than the value 13,000 found by Reynolds.

61

AS Re $\uparrow$ MORE "MIXING"

STRONGER DISTURBANCES

MORE ABOUT CONSTITUTIVE RELATIONS FOR VISCOSITY

NEWTONIAN:

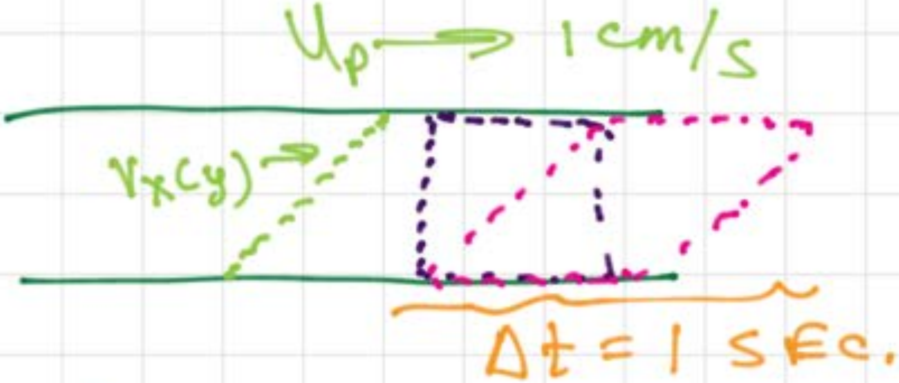
$$\tau_{yx} = \mu \frac{\partial v_x}{\partial y}$$

$$= \mu \, \dot{\gamma}_x$$

SHEAR RATE

$\dot{\gamma}_x$ HAS UNITS OF $(\text{TIME})^{-1}$

IT IS NORMALIZED RATE OF DEFORMATION



$$\dot{\gamma} = \frac{\partial v_x}{\partial y} = \frac{1}{s}$$

ORIGINAL SHAPE IS DISLACED BY ITS SIZE IN 1 S.

OTHER RELATIONS EXIST.

BINGHAM PLASTIC

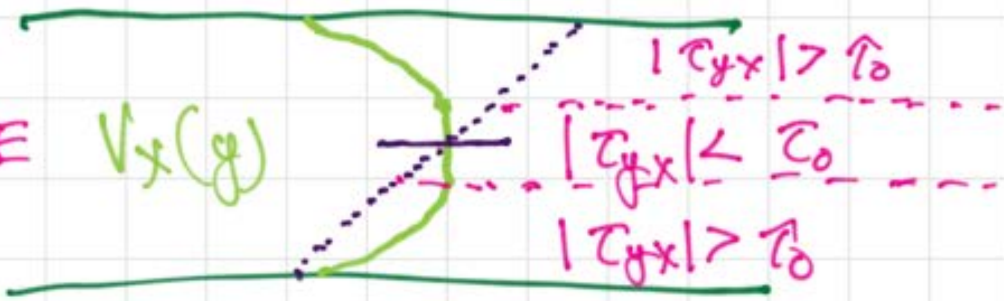
$$|\tau_{yx}| < \tau_0 \quad \dot{\gamma}_x = 0$$

<u>NO FLOW</u>: FRESH PAINT ON A WALL

$$|\tau_{yx}| > \tau_0$$

$$\tau_{yx} = \pm \tau_0 + \mu_0 \dot{\gamma}_x$$

FLAT PROFILE IN MIDDLE



rest file format.

Figure 2.17 Relationship between the shear stress and shear rate for Newtonian and non-Newtonian fluids. Fluid properties are Newtonian fluid, $\mu = 0.01$ g cm^{-1} s^{-1}; Bingham plastic, $\tau_0 = 0.01$ dyn cm^{-2}, $\mu_0 = 0.01$ g cm^{-1} s^{-1}; shear-thickening fluid (dilatant), $m = 0.01$ g cm^{-1}, $n = 2.0$; and shear thinning fluid (pseudoplastic), $m = 0.01$ g cm^{-1} s$^{-1.5}$, $n = 0.5$.

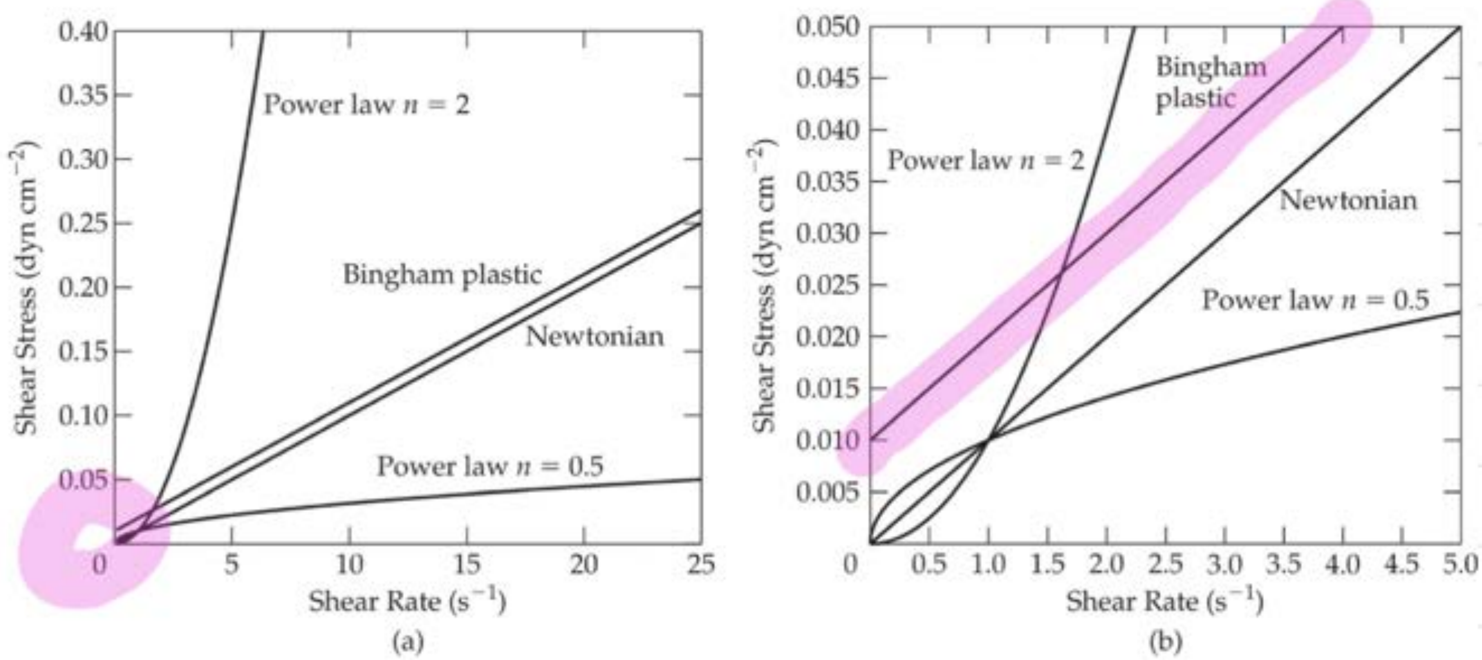
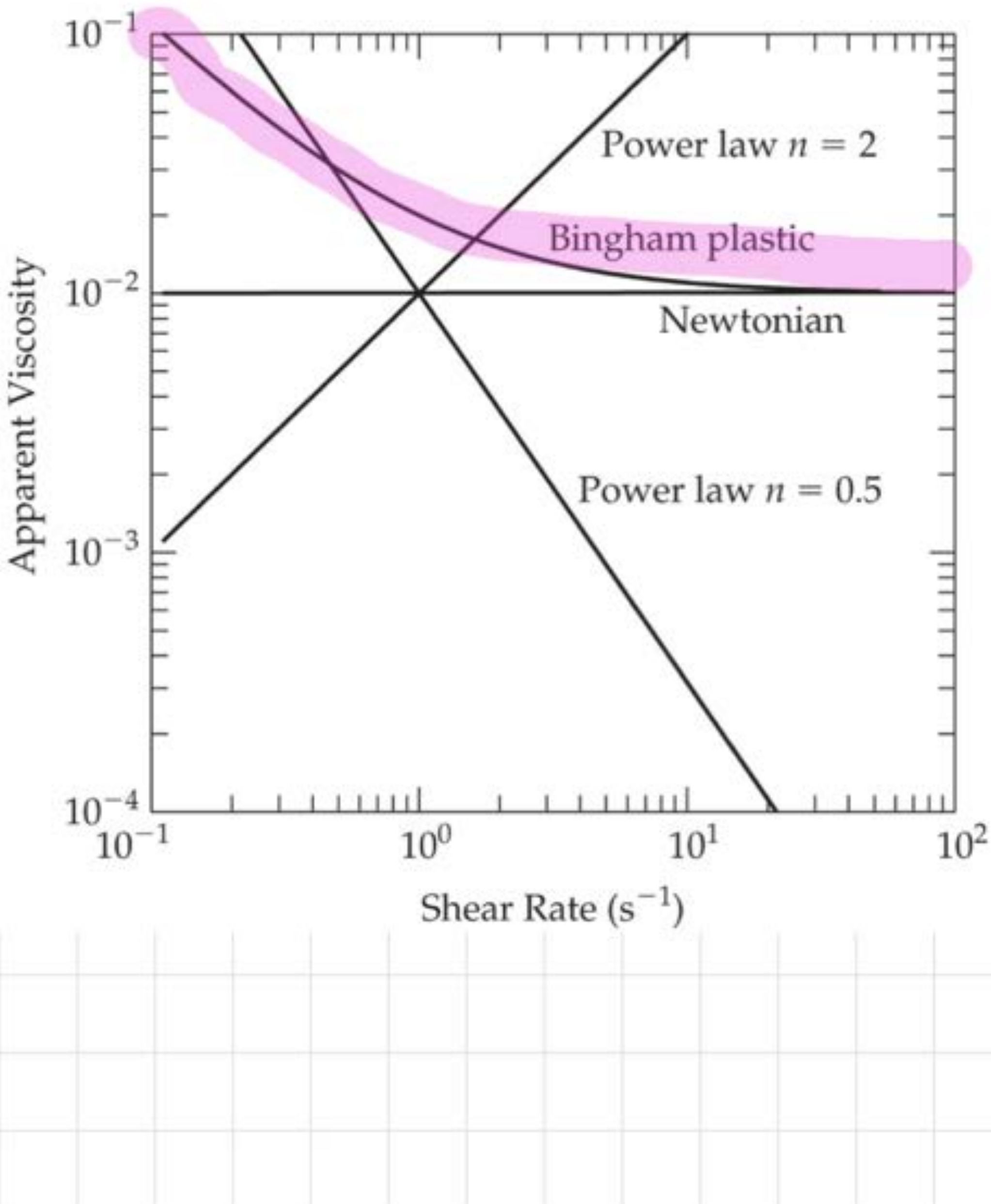


Figure 2.18 Apparent viscosity versus shear rate for fluids shown in Figure 2.17.



$$0 = \frac{\partial \tau_{yx}}{\partial y} + \rho g_x$$

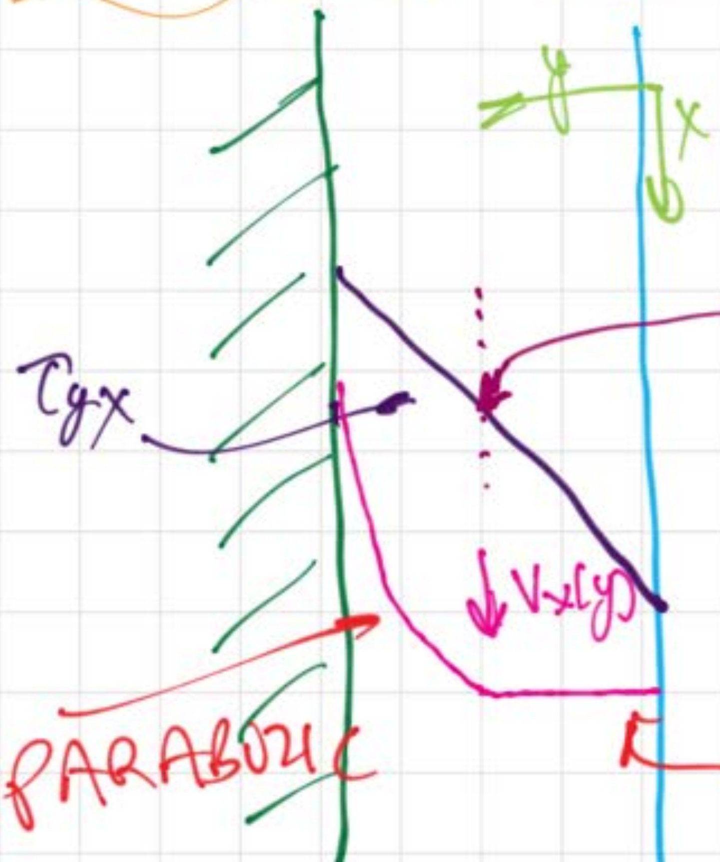
$$\tau_{yx} = -\rho g_x y$$

$$\tau_{yx}(h) = -\rho g_x h$$

IF $|\tau_{yx}(h)| < \tau_0$

$$|\tau_{yx}(h)| > \tau_0$$

$$\tau_{yx}(y) = \tau_0$$

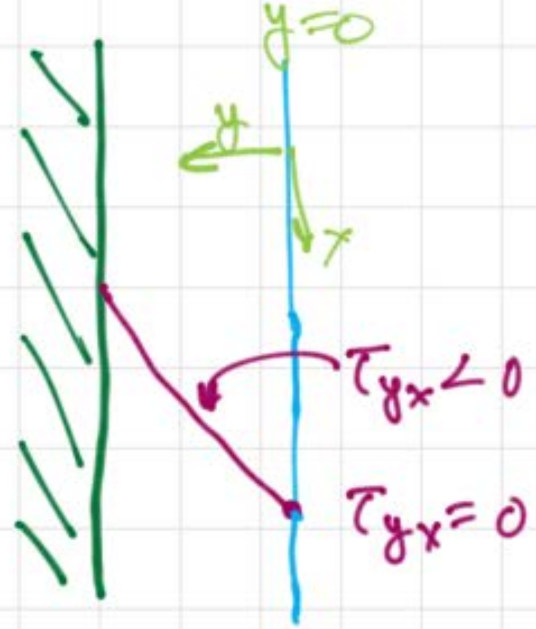


FLAT

$$\tau_{yx} = -\rho g_x y$$

FOR A BINGHAM PLASTIC:

$$\tau_{yx} = -\tau_0 + \mu \frac{\partial v_x}{\partial y}$$



SO y IS POSITIVE INWARD

SUBS CONSTITUTIVE RELATION INTO INTEGRATED MOMENTUM EQ.

$$-\tau_0 + \mu \frac{\partial v_x}{\partial y} = -\rho g_x y$$

$$\mu \frac{\partial v_x}{\partial y} = -\rho g_x y + \tau_0$$

$$\int dv_x = \int \left(\frac{-\rho g_x y}{\mu} + \tau_0 \right) dy$$

$$v_x = \frac{-\rho g_x}{\mu} \frac{y^2}{2} + \frac{\tau_0}{\mu} y + C_1$$

$$v_x(h) = 0$$

$$\therefore \quad C_1 = \frac{\rho g_x}{\mu} \frac{h^2}{2} - \frac{\tau_0}{\mu} h$$

$$\therefore \quad v_x = \frac{\rho g_x}{2\mu}\left(h^2 - y^2\right) + \frac{\tau_0}{\mu}(y - h)$$

VALID IN $|\tau_{yx}| > |\tau_0|$

FOR $\tau_{yx} \leq \tau_0$

$$v_x = v_{MAX}$$

HOW DO WE FIND $y_0 \Rightarrow (\tau_{yx} = \tau_0)$?

FIND $\frac{dv_x}{dy} = 0$

$$\frac{dv_x}{dy} = -\frac{\rho g_x}{\mu} y + \frac{\tau_0}{\mu} = 0$$

$$y_0 \equiv y\Big|_{\tau=\tau_0} = \frac{\tau_0}{\rho g_x}$$

SO FOR $y < y_0$

$$v_x = \frac{\rho g_x}{2\mu}\left(h^2 - y^2\right) + \frac{\tau_0}{\mu}(y-h)$$

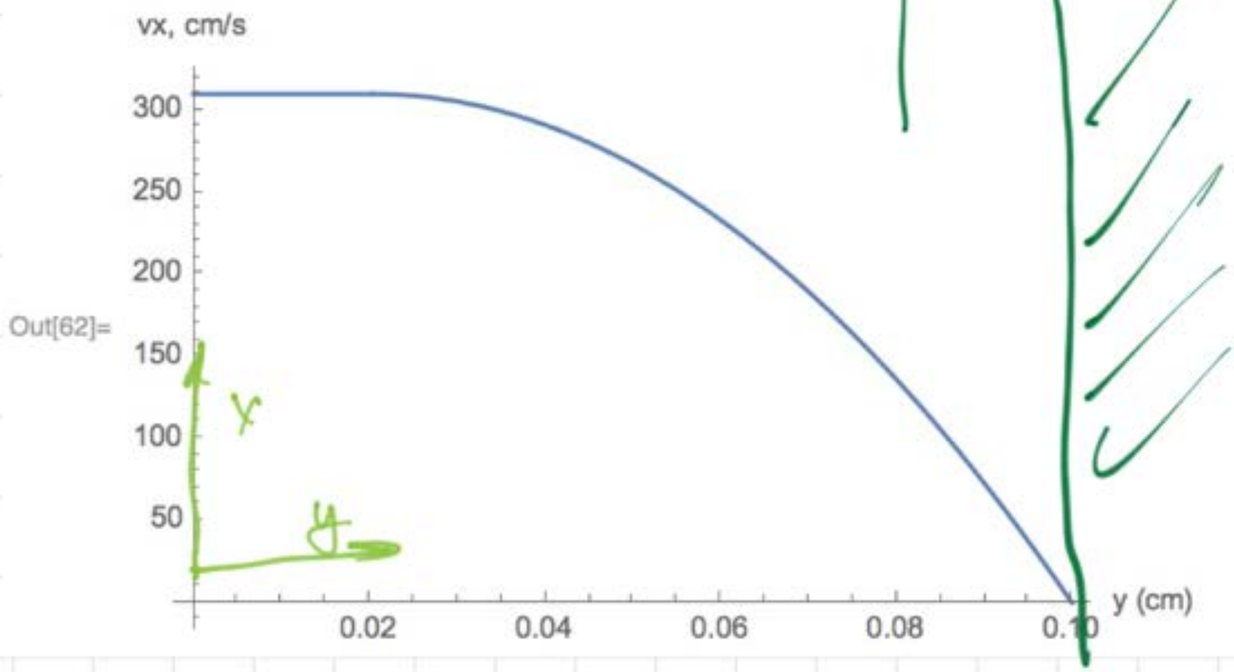
$$v_{x\,MAX} = \frac{\rho g_x}{2\mu}\left(h^2 - \left(\frac{\tau_0}{\rho g_x}\right)^2\right) + \frac{\tau_0}{\mu}\left(\frac{\tau_0}{\rho g_x} - h\right)$$

$$v_{MAX} = \frac{\left(\tau_0 - \rho g_x h\right)^2}{2 g_x \mu \rho}$$

DOWNWARD

In[62]:= `Show[%, %%, PlotRange -> All]`

Out[62]=




```
Plot[veex /. {gx → 980, ρ → 1, h → .1, τ0 → 20, μ → .01}, {y, 20 / 980, .1},
  AxesLabel → {"y (cm)", "vx, cm/s"}]
```

TO GET FLOWRATE INTEGRATE 2 PIECES

$$Q = W \int_0^{\tau_0/\rho g_x} V_{MAX}\, dy +$$

$$W \int_{\tau_0/\rho g_x}^{h} \left(\frac{\rho g_x}{2\mu}\left(h^2 - y^2\right) + \frac{\tau_0}{\mu}(y-h) \right) dy$$

$$Q = W \frac{(\tau_0 - \rho g_x h)^2 (\tau_0 + 2\rho g_x h)}{6 g_x^2 \mu \rho^2}$$

IF WE DO A SERIES EXPANSION

$$Q = \frac{\rho g_x h^3}{3\mu} - \frac{\tau_0 h^2}{2\mu} + O(\tau_0^3)$$

EXTENT OF REDUCTION IN FLOW

POWER LAW FLUIDS

APPARENT VISCOSITY CHANGES WITH SHEAR RATE

$$\eta_{APP} = m \left| \dot{\gamma} \right|^{n-1}$$

$n = 1$ NEWTONIAN

$n < 1$ SHEAR THINNING

$n > 1$ SHEAR THICKENING

rest file format.

Figure 2.17 Relationship between the shear stress and shear rate for Newtonian and non-Newtonian fluids. Fluid properties are Newtonian fluid, $\mu = 0.01$ g cm^{-1} s^{-1}; Bingham plastic, $\tau_0 = 0.01$ dyn cm^{-2}, $\mu_0 = 0.01$ g cm^{-1} s^{-1}; shear-thickening fluid (dilatant), $m = 0.01$ g cm^{-1}, $n = 2.0$; and shear thinning fluid (pseudoplastic), $m = 0.01$ g cm^{-1} s$^{-1.5}$, $n = 0.5$.

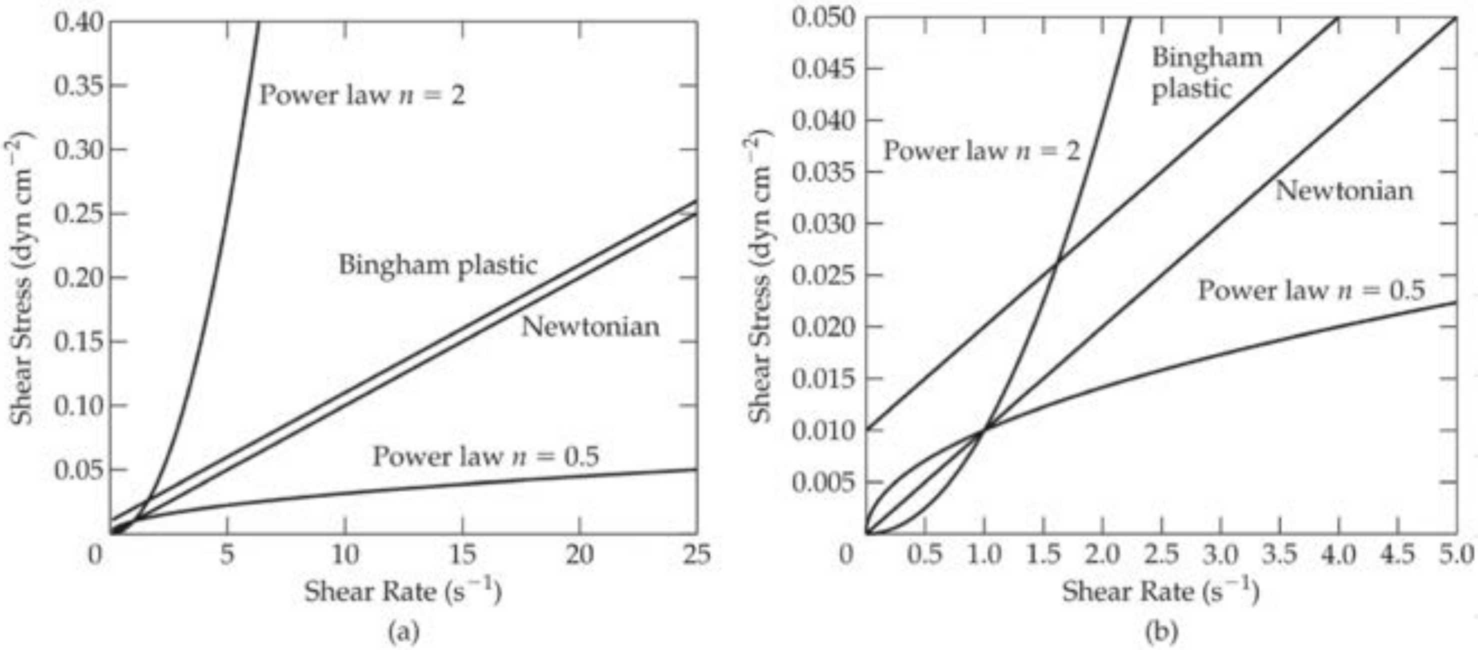
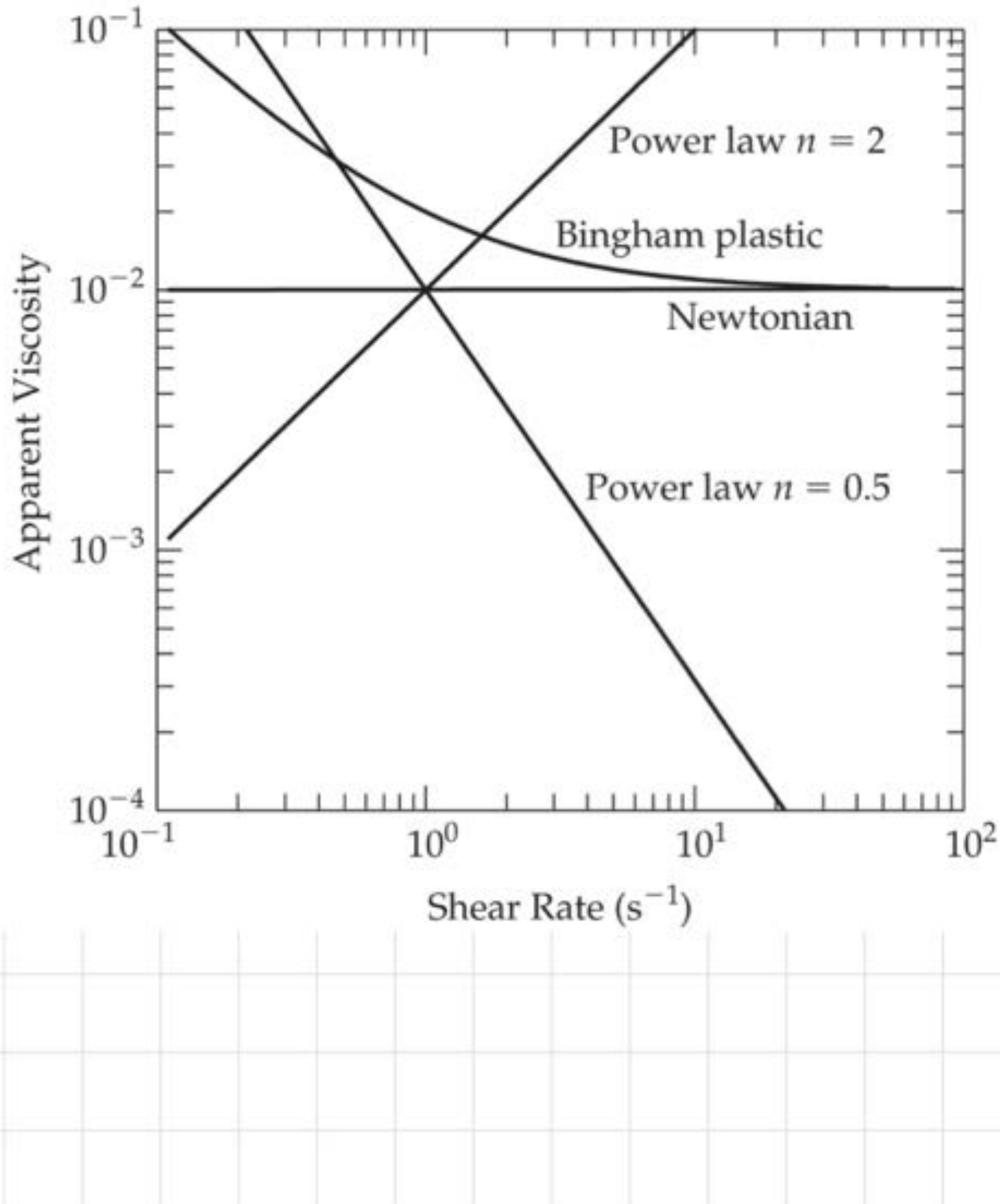


Figure 2.18 Apparent viscosity versus shear rate for fluids shown in Figure 2.17.



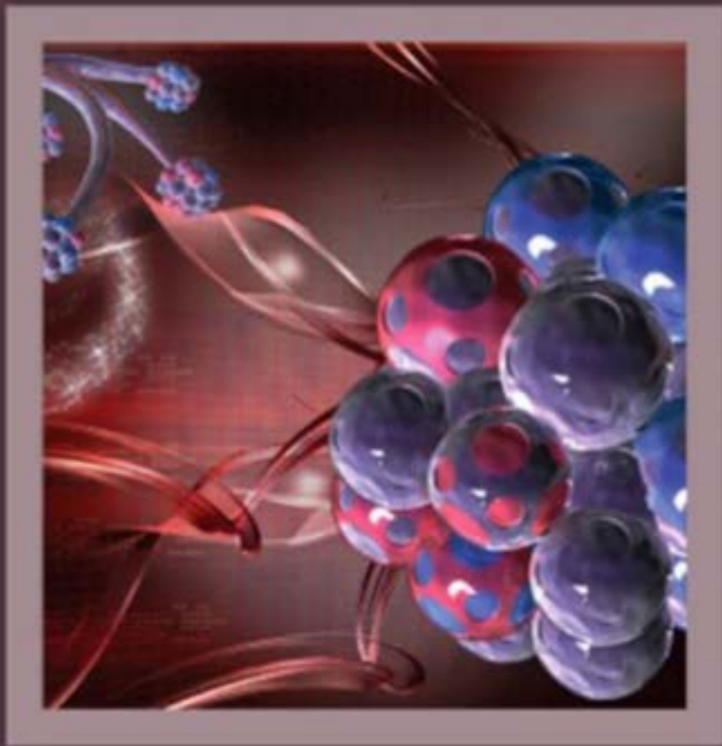
BLOOD FLOW

In the largest arteries for humans, the Reynolds number is large enough that we would expect turbulent flow and for the first few generations of the arterial system the flow is pulsatile.

So there are some limits to exact application of the analysis that we can do.

However, we can get some useful insights so we shall proceed. (You can read more at another time.)

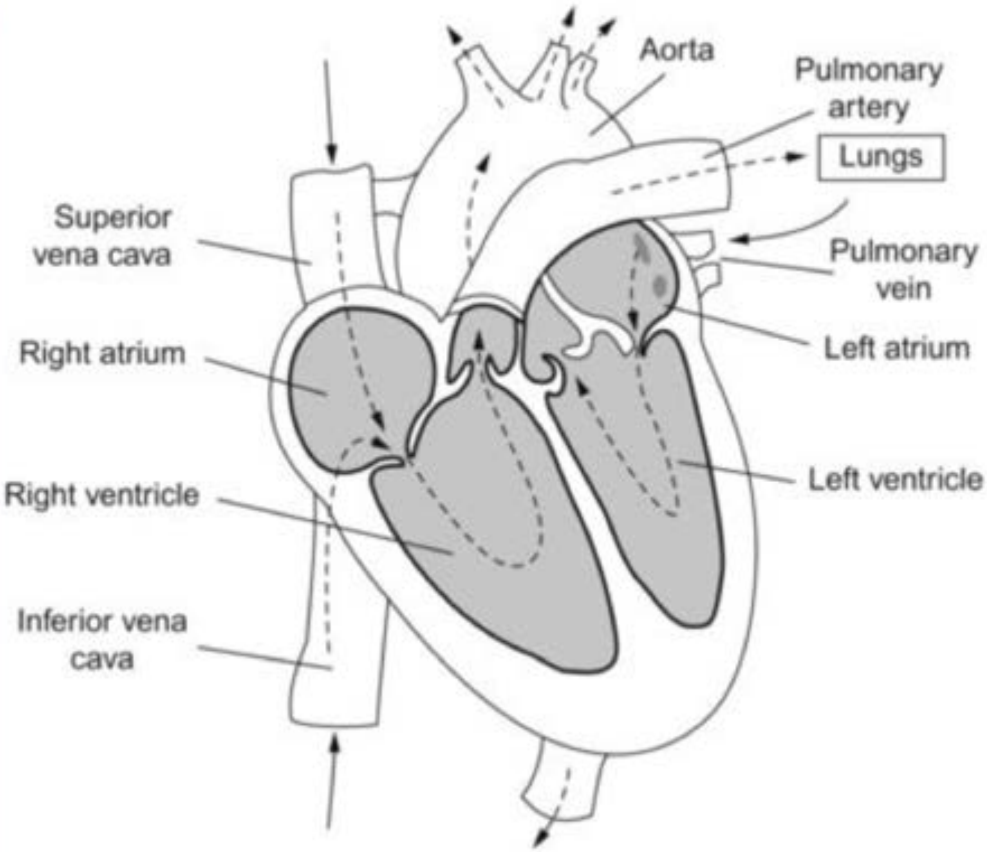
ACADEMIC PRESS SERIES IN BIOMEDICAL ENGINEERING

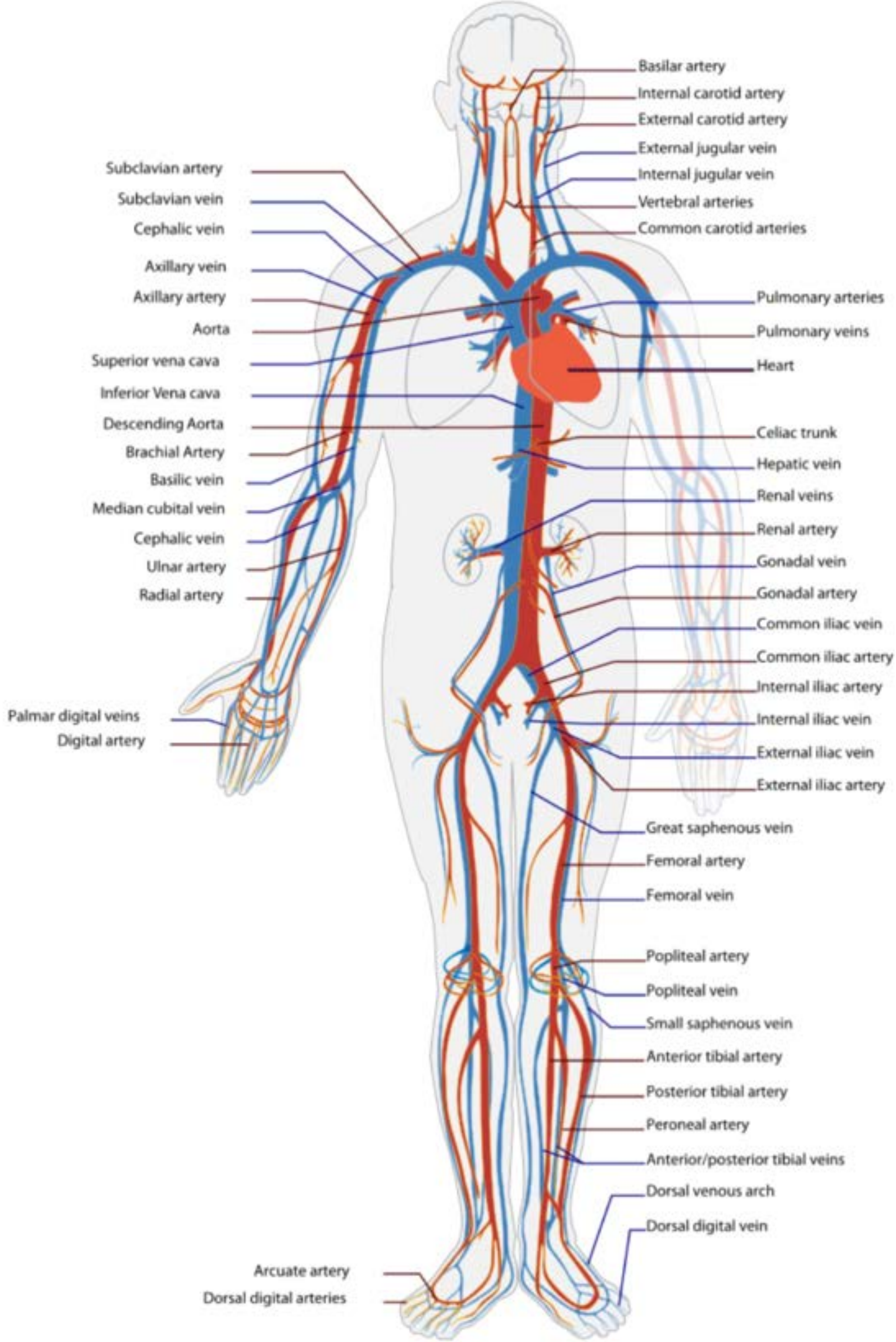


BIOFLUID MECHANICS

AN INTRODUCTION TO FLUID MECHANICS, MACROCIRCULATION, AND MICROCIRCULATION

DAVID A. RUBENSTEIN
WEI YIN
MARY D. FRAME


Aorta
Pulmonary artery
Lungs
Superior vena cava
Pulmonary vein
Right atrium
Left atrium
Left ventricle
Right ventricle
Inferior vena cava


Basilar artery
Internal carotid artery
External carotid artery
External jugular vein
Internal jugular vein
Vertebral arteries
Common carotid arteries
Subclavian artery
Subclavian vein
Cephalic vein
Axillary vein
Axillary artery
Aorta
Superior vena cava
Inferior Vena cava
Descending Aorta
Brachial Artery
Basilic vein
Median cubital vein
Cephalic vein
Ulnar artery
Radial artery
Pulmonary arteries
Pulmonary veins
Heart
Celiac trunk
Hepatic vein
Renal veins
Renal artery
Gonadal vein
Gonadal artery
Common iliac vein
Common iliac artery
Internal iliac artery
Internal iliac vein
External iliac vein
External iliac artery
Palmar digital veins
Digital artery
Great saphenous vein
Femoral artery
Femoral vein
Popliteal artery
Popliteal vein
Small saphenous vein
Anterior tibial artery
Posterior tibial artery
Peroneal artery
Anterior/posterior tibial veins
Dorsal venous arch
Dorsal digital vein
Arcuate artery
Dorsal digital arteries