CBE 30357
9/12/17

TOPICS FOR TODAY

1) REVIEW OF SLIDING SURFACE FLOW

2) FLOWS CAUSED BY A PRESSURE GRADIENT

3) FLOWS CAUSED BY GRAVITY

* ALONG WITH "WIND", THE PRIMARY ACTIONS OR FORCES THAT CAUSE FLUID FLOW.

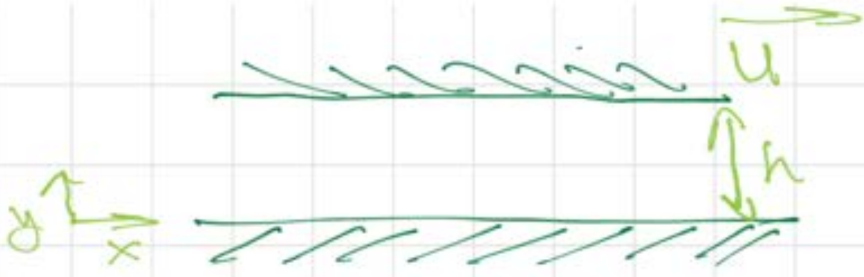


TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x \frac{\partial v_x}{\partial x}} + \cancel{v_y \frac{\partial v_x}{\partial y}} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial x}} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \cancel{\rho g_x}$$

y direction

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x \frac{\partial v_y}{\partial x}} + \cancel{v_y \frac{\partial v_y}{\partial y}} + \cancel{v_z \frac{\partial v_y}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \mu\left[\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho g_y$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x \frac{\partial v_z}{\partial x}} + \cancel{v_y \frac{\partial v_z}{\partial y}} + \cancel{v_z \frac{\partial v_z}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial z}} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \cancel{\rho g_z}$$

AFTER SOME WORK

$$0 = \mu \frac{\partial^2 v_x}{\partial y^2}$$

BOUNDARY CONDITIONS..

$$v_x(0) = 0, \quad v_x(h) = U$$

FLUID STICKS TO SOLID SURFACES!

$$v_x = C_1 y + C_2$$

$$v_x(y) = \frac{y}{h} U$$

WHICH STRESS COMPONENTS ARE IMPORTANT ??

$$V_x(y) = \frac{y}{h} U$$ ← NO VISCOSITY

CONSTANT, NOT A FUNCTION OF Y

$$\tau_{yx} = \mu \frac{\partial v_x}{\partial y} = \mu \frac{U}{h}$$ ← VISCOSITY IN STRESS

(FORCE/AREA)

$$FORCE_x = \tau_{yx}\ W \times L$$

WIDTH IN Z

LENGTH IN X

$$= \mu \frac{U}{h} W L$$

FASTER ↑ F

MORE VISCOUS ↑ F

CLOSER SPACING ↑ F

$$\text{AVERAGE VELOCITY} = \frac{\text{TOTAL VOLUMETRIC FLOW}}{\text{AREA OF FLOW}}$$

$$\langle V_x \rangle = \frac{\int_0^W \int_0^h v_x(y)\, dy\, dz}{\int_0^W \int_0^h dy\, dz}$$

$$\langle V_x \rangle = \frac{W \frac{U}{h} \left. \frac{y^2}{2} \right|_0^h}{W h} = \frac{U}{2}$$

SUPPOSE WE KEEP THE FLOW GEOMETRY THE SAME, BUT CAUSE THE FLOW BY A PRESSURE GRADIENT

$P_1 > P_2$

NO CHANGE W/Z

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x \frac{\partial v_x}{\partial x}} + \cancel{v_y \frac{\partial v_x}{\partial y}} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \cancel{\rho g_x}$$

y direction

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x \frac{\partial v_y}{\partial x}} + v_y \frac{\partial v_y}{\partial y} + \cancel{v_z \frac{\partial v_y}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial y}} + \mu\left[\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \cancel{\rho g_y}$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x \frac{\partial v_z}{\partial x}} + \cancel{v_y \frac{\partial v_z}{\partial y}} + \cancel{v_z \frac{\partial v_z}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial z}} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \cancel{\rho g_z}$$

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 V_x}{\partial y^2}$$

BOUNDARY CONDITIONS:

$$V_x(h) = 0$$
$$V_x(-h) = 0$$

[NO SLIP]

WE SOLVE:

$$\frac{d^2 V_x}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$$

A subtle and perhaps confusing argument will now be given! 😳

We conclude that V_x does not change in x.

We conclude that $\frac{\partial P}{\partial x}$ does not change with y

Hence, the only way for these terms to be always equal is if they are equal to a constant.

$$\frac{d^2 V_x}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} = \text{CONSTANT}$$

NOW WE PROCEED...

$$\frac{d}{dy}\left(\frac{dV_x}{dy}\right) = \frac{1}{\mu}\frac{dP}{dx}$$

$$d\left(\frac{dV_x}{dy}\right) = \frac{1}{\mu}\frac{dP}{dx}\,dy$$

$$\int d\left(\frac{dV_x}{dy}\right) = \int \frac{1}{\mu}\frac{dP}{dx}\,dy$$

$$\frac{dv_x}{dy} = \frac{1}{\mu}\frac{dP}{dx}y + C_1$$

$$dv_x = \left(\frac{1}{\mu}\frac{dP}{dx}y + C_1\right)dy$$

$$\int dv_x = \int\left(\frac{1}{\mu}\frac{dP}{dx}y + C_1\right)dy$$

$$v_x = \frac{1}{\mu}\frac{dP}{dx}\frac{y^2}{2} + C_1 y + C_2$$

NOW WE NEED TO APPLY BOUNDARY CONDITIONS

$$v_x(h) = 0 = \frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2} + C_1 h + C_2$$

$$v_x(-h) = 0 = \frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2} - C_1 h + C_2$$

$$\therefore C_1 = 0$$

$$C_2 = -\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}$$

WHICH GIVES

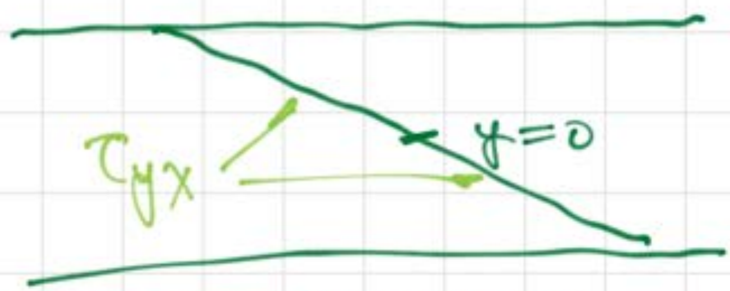
$$V_x(y) = -\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}\left(1 - \frac{y^2}{h^2}\right)$$

SHEAR STRESS

$$\tau_{yx} = \mu \frac{\partial V_x}{\partial y}$$

$$= \frac{dP}{dx} y$$

$(-)$



STRESS IS MAXIMUM AT WALLS, 0 IN MIDDLE

WHAT IS AVERAGE VELOCITY?

$$\langle v_x \rangle = \frac{\int_0^w \int_{-h}^{h} v_x(y)\,dy\,dz}{\int_0^w \int_{-h}^{h} dy\,dz}$$

$$v_x(y) = -\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}\left(1-\frac{y^2}{h^2}\right)$$

$$\langle v_x \rangle = \frac{-\frac{1}{\mu}\frac{dP}{dx}\frac{h^2}{2}w}{2wh}\int_{-h}^{h}\left(1-\frac{y^2}{h^2}\right)dy$$

$$= \frac{-\frac{1}{\mu}\frac{dP}{dx}\frac{h^{\cancel{2}}\cancel{w}}{2}}{2\cancel{w}\cancel{h}}\left(y-\frac{y^3}{3h^2}\right)\Bigg|_{-h}^{h}$$

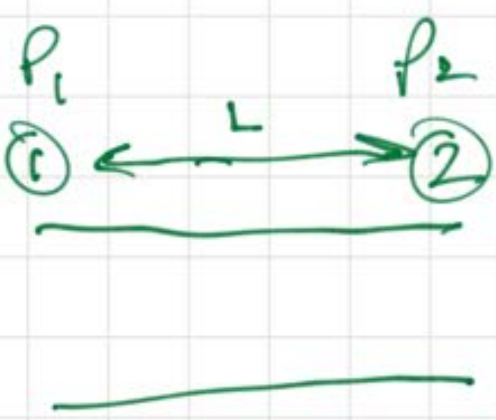
$$= \frac{-1}{4\mu}\frac{dP}{dx}h\left(\underbrace{\left(h-\frac{1}{3}h\right)}_{2/3\,h}-\underbrace{\left(-h+\frac{1}{3}h\right)}_{+2/3\,h}\right)$$

$$\langle v_x \rangle = \frac{1}{3\mu}\left(-\frac{dP}{dx}\right) h^2$$

PRESSURE "DROPS" LINEARLY WITH DISTANCE, HENCE

$$\frac{dP}{dx} < 0$$

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} = \text{CONSTANT}$$



$$\frac{dP}{dx} = C_1$$

$$dP = C_1 dx$$

$$\int dP = \int C_1 dx$$

$$P = C_1 x + C_2$$

@ $x=0$, $P=P_1$

$$\therefore \quad P = C_1 x + P_1$$

IF $P = P_2$ @ $x = L$

$$P_2 = C_1 L + P_1$$

$$C_1 = \frac{P_2 - P_1}{L}$$

OR

$$P = \frac{P_2 - P_1}{L} x + P_1$$

$$= \frac{dP}{dx} < 0$$

PRESSURE DECREASES LINEARLY WITH DISTANCE

How does gravity cause fluid flow?

X-DIRECTION NAVIER-STOKES EQ.

$$0 = \mu \frac{d^2 V_x}{dy^2} + \rho g \cos\theta$$

<u>BOUNDARY CONDITIONS</u>

$$V_x(0) = 0$$

AT $y = h$ WE HAVE TO FIND SOMETHING ELSE....

$$\tau_{yx}(y=h) = 0$$ NO <u>STRESS</u> ☎

$$0 = \mu \frac{d^2 v_x}{dy^2} + \rho g \cos\theta$$

$$\frac{d^2 v_x}{dy^2} = -\frac{\rho g}{\mu} \cos\theta$$

$$\frac{d}{dy}\left(\frac{dv_x}{dy}\right) = -\frac{\rho g}{\mu} \cos\theta$$

$$\int d\left(\frac{dv_x}{dy}\right) = -\frac{\rho g}{\mu} \cos\theta \int dy$$

$$\frac{dv_x}{dy} = -\frac{\rho g}{\mu} \cos\theta \, y + C_1$$

$$\int dv_x = \int\left(-\frac{\rho g}{\mu} \cos\theta \, y + C_1\right) dy$$

$$v_x = -\frac{\rho g}{\mu} \cos\theta \frac{y^2}{2} + C_1 y + C_2$$

@ $y=0$ $V_x(0)=0$

$\therefore C_2 = 0$

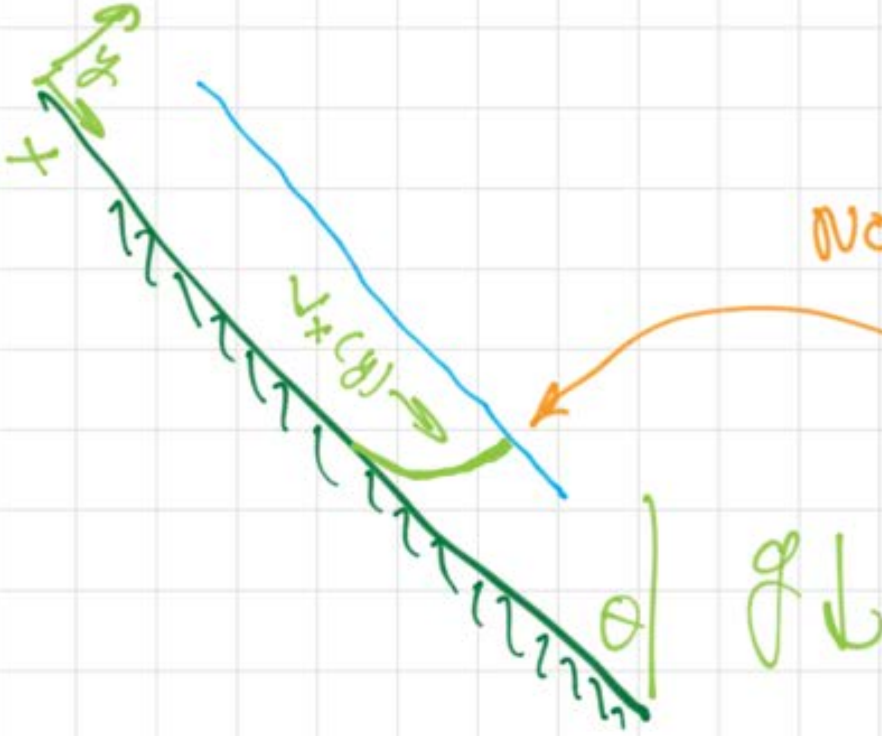
@ $y=h$ $\frac{dV_x}{dy}=0$

$$\frac{dV_x}{dy} = \frac{-\rho g}{\mu}\cos\theta\, y + C_1$$

$$\therefore C_1 = \frac{\rho g}{\mu}\cos\theta\, h$$

THUS THE PROFILE IS

$$V_x(y) = \frac{\rho g}{\mu}\cos\theta\left(\frac{y^2}{2} - yh\right)$$



NO STRESS

$$\left.\frac{dv_x}{dy}\right|_{y=h} = 0$$

The mechanism of exercise-induced asthma is . . .

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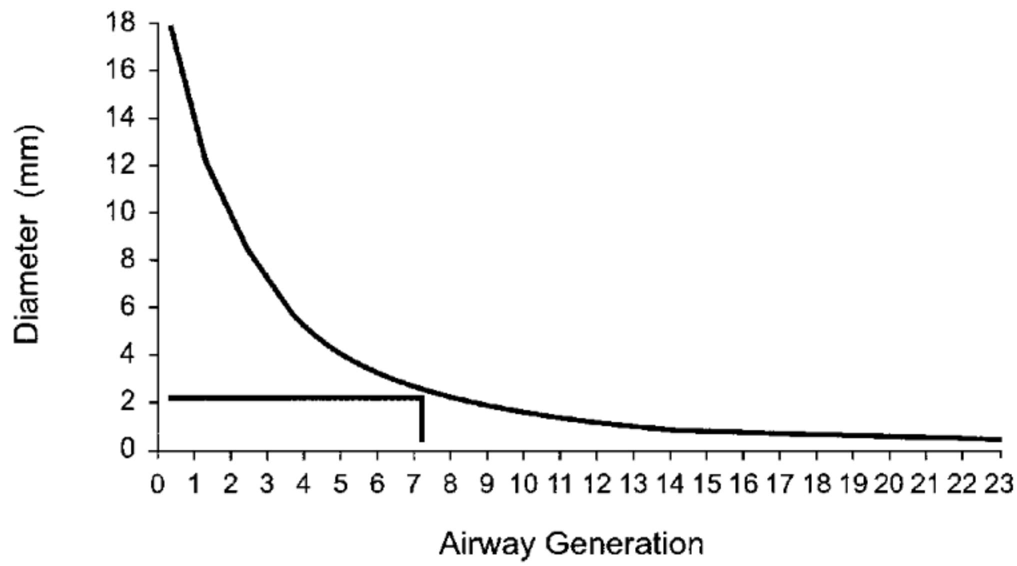
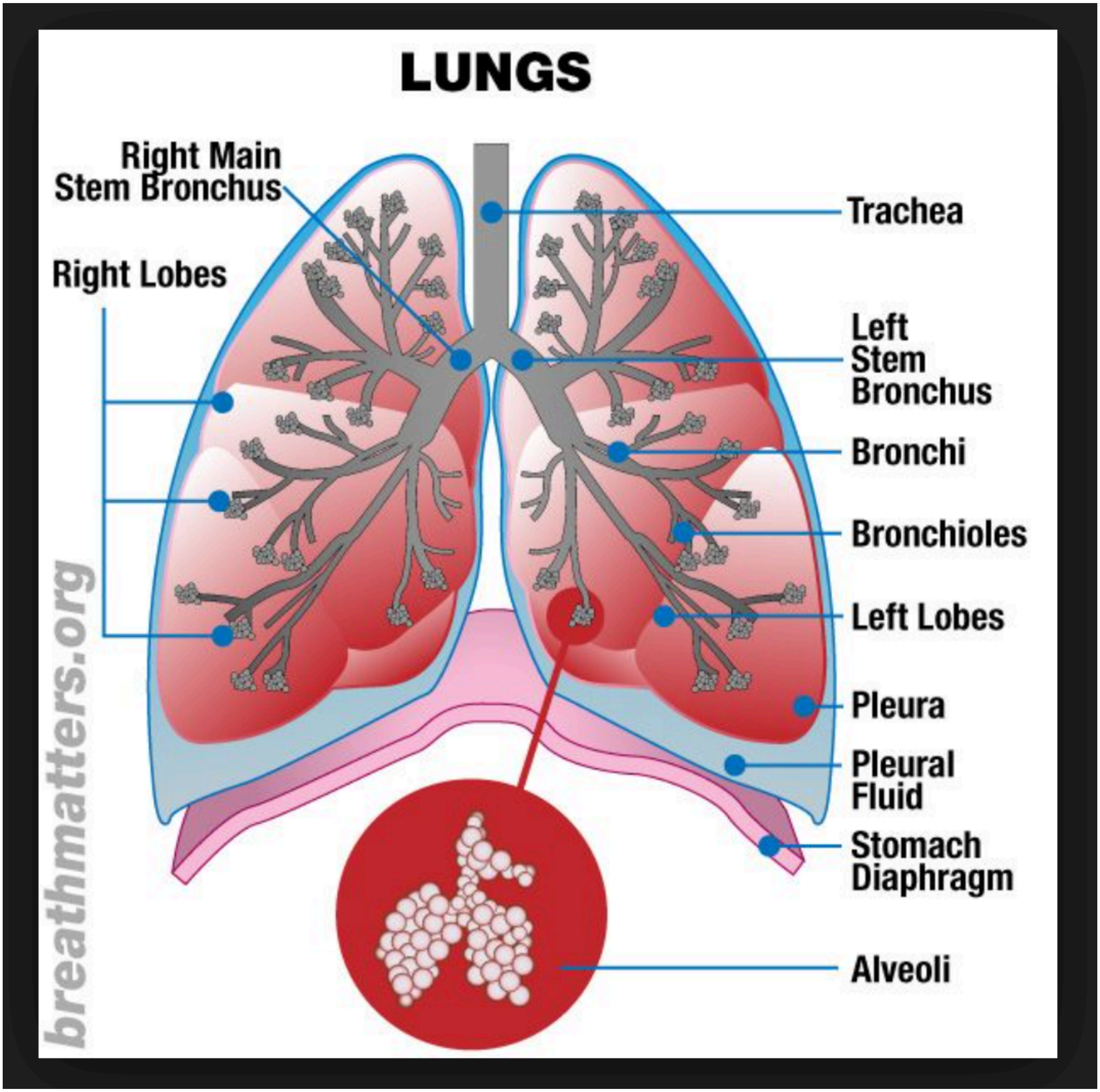


Figure 1. *Airway diameters (mm) drawn against airway generations. (Adapted with permission from Ref. 6.)*

MECHANISMS OF REMODELLING

perhaps best illustrated by irritant-induced asthma. Certain characteristic changes, such as growth of ASM, seem likely to contribute to airways hyperresponsiveness (AHR), but the consequences of other changes, such as subepithelial fibrosis, are not so intuitively obvious. Even the link between growth of ASM and AHR is deduced largely on the basis of modelling studies. One of the major challenges facing the field is to devise strategies that link structure and function directly. For

Airway smooth muscle (remodeling)


LUNGS
Right Main Stem Bronchus
Right Lobes
Trachea
Left Stem Bronchus
Bronchi
Bronchioles
Left Lobes
Pleura
Pleural Fluid
Stomach Diaphragm
Alveoli
breathmatters.org

Suppose we want to consider the flow in a trachea.

The geometry is a cylinder.

We need the Navier-Stokes equations in cylindrical coordinates.

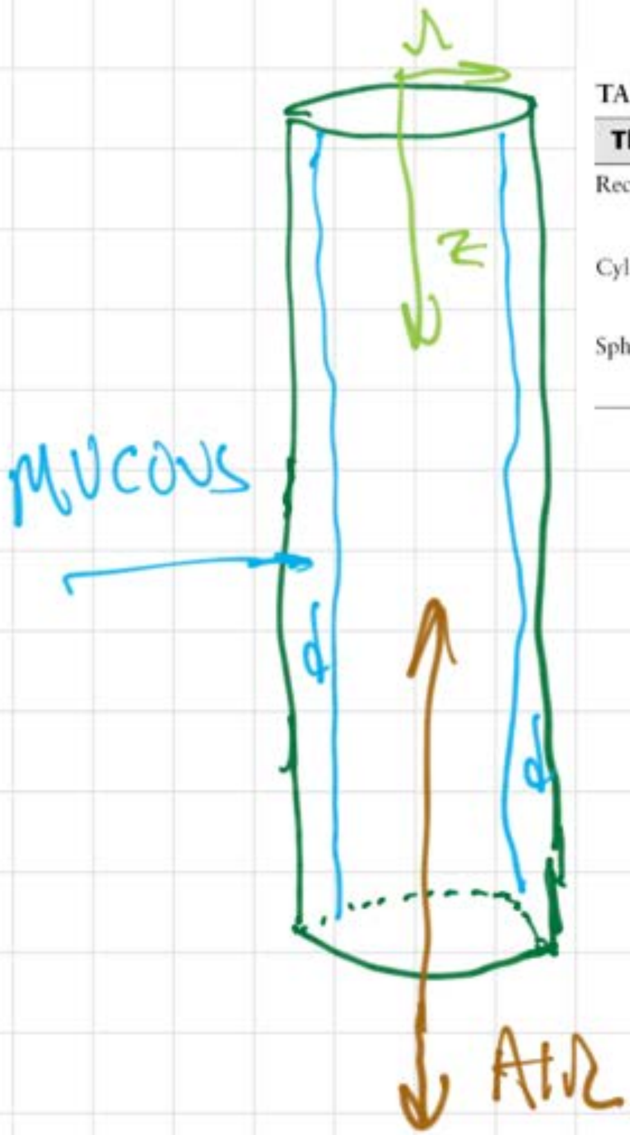


TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}\right)$

$$\frac{\partial v_z}{\partial z} = 0$$

$$v_z = v_z(r)$$

ONLY

Cylindrical coordinates

r direction

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + v_z\cancel{\frac{\partial v_r}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial r}}^{\,0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cancel{\frac{\partial(r v_r)}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right] + \cancel{\rho g_r} \qquad (3.3.27a)$$

θ direction

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + v_r\cancel{\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}^{\,=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cancel{\frac{\partial(r v_\theta)}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancel{\rho g_\theta} \qquad (3.3.27b)$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_z}{\cancel{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial z}} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \qquad (3.3.27c)$$

$$0 = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z$$

FOR LIQUID FILM

$$r = R \quad v_z(R) = 0$$

$$r = R - h \quad \left.\frac{\partial v_z}{\partial r}\right|_{r=(R-h)} = 0$$

SEEMS LIKE ESSENTIALLY SAME PROBLEM IF $\frac{h}{R} \ll 1$

$$0 = \mu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) + \rho g_z$$

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{d v_z}{dr} \right) = -\frac{\rho g_z}{\mu}$$

$$\frac{d}{dr}\left(r \frac{d v_z}{dr} \right) = -\frac{\rho g_z}{\mu} r$$

$$\int d\left(r \frac{d v_z}{dr} \right) = \int \frac{\rho g_z}{\mu} r \, dr$$

$$r \frac{d v_z}{dr} = -\frac{\rho g_z}{\mu} \frac{r^2}{2} + C_1$$

$$\frac{d v_z}{dr} = -\frac{\rho g_z}{\mu} \frac{r}{2} + \frac{C_1}{r}$$

$$\frac{dv_z}{dr} = \frac{-\rho g_z}{\mu}\frac{r}{2} + \frac{C_1}{r}$$

$$\int dv_z = \int \left(\frac{-\rho g_z}{\mu}\frac{r}{2} + \frac{C_1}{r} \right) dr$$

$$v_z = \frac{-\rho g_z}{4\mu} r^2 + C_1 \ln r + C_2$$

FIT BC'S

$$v_z(r) = \frac{\rho g_z}{4\mu}\left((R^2 - r^2) + 2(h-R)^2 \ln\left(\frac{r}{R}\right) \right)$$

IF FLOW DOMAIN INCLUDES $r = 0$, THE "ln" TERM MUST BE EXCLUDED.

USING MATHEMATICA, THE AVERAGE VELOCITY IS EASILY OBTAINED

Find the average velocity

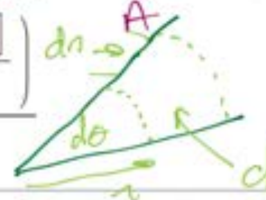
In[9]:=
```
Integrate[vzee r, {r, R, R - h}, Assumptions → {h > 0, h < R}] /
 Integrate[ r, {r, R, R - h}, Assumptions → {h > 0, h < R}]
```

Out[9]=
$$\frac{gz\,\rho\left(-\frac{1}{4}h\,(h-2R)\left(3h^2-6hR+2R^2\right)+(h-R)^4\,\mathrm{Log}\left[1-\frac{h}{R}\right]\right)}{4\left(-\frac{R^2}{2}+\frac{1}{2}(-h+R)^2\right)\mu}$$

In[10]:=
```
FullSimplify[%]
```

$$Q = \int V_z(r,\theta)\, dA$$

Out[10]=
$$\frac{gz\,\rho\left(-3h^2+6hR-2R^2+\frac{4\,(h-R)^4\,\mathrm{Log}\left[1-\frac{h}{R}\right]}{h\,(h-2R)}\right)}{8\mu}$$



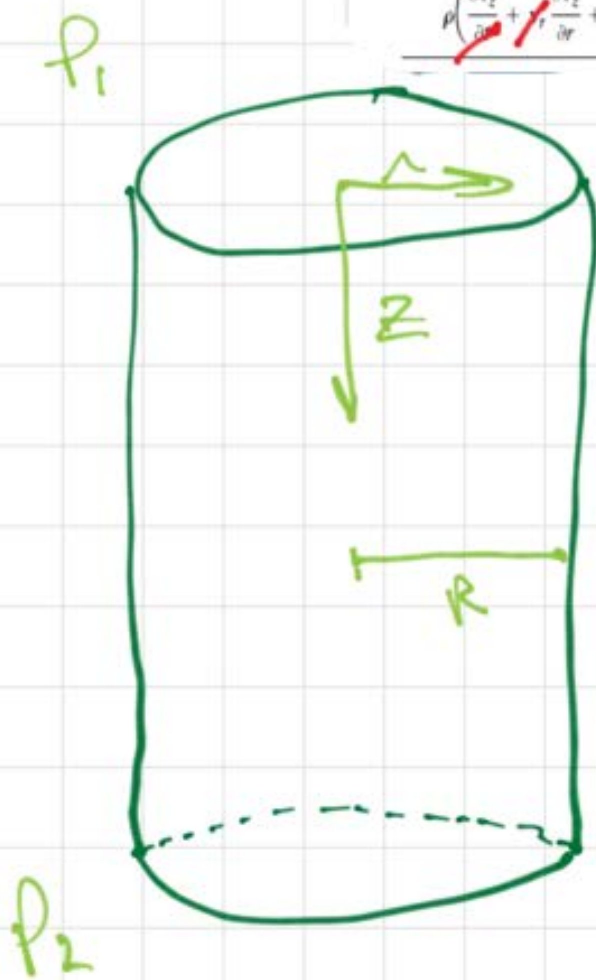
$$dA = (r\,d\theta)(dr)$$

$$\frac{\int_R^{R-h} V_z(r)\, r\, dr}{\int_R^{R-h} r\, dr} = \langle V_z \rangle$$

Suppose that we have a circular pipe oriented,vertically.

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \frac{\cancel{v_\theta}}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$



$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z$$

BC's

$$v_z(R) = 0$$

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=0} = 0$$

"SYMMETRY"

2 CASES:

1) OPEN AT BOTH ENDS $P_1 = P_2$

2) CLOSE BOTTOM: NO FLOW

LET'S DO CASE #2

$$0 = -\frac{\partial P}{\partial z} + \rho g_z$$

$$\frac{dP}{dz} = \rho g_z$$

$$\int dP = \int \rho g_z \, dz$$

$$P = \rho g_z z + C_0$$

$P(0) = P_1$

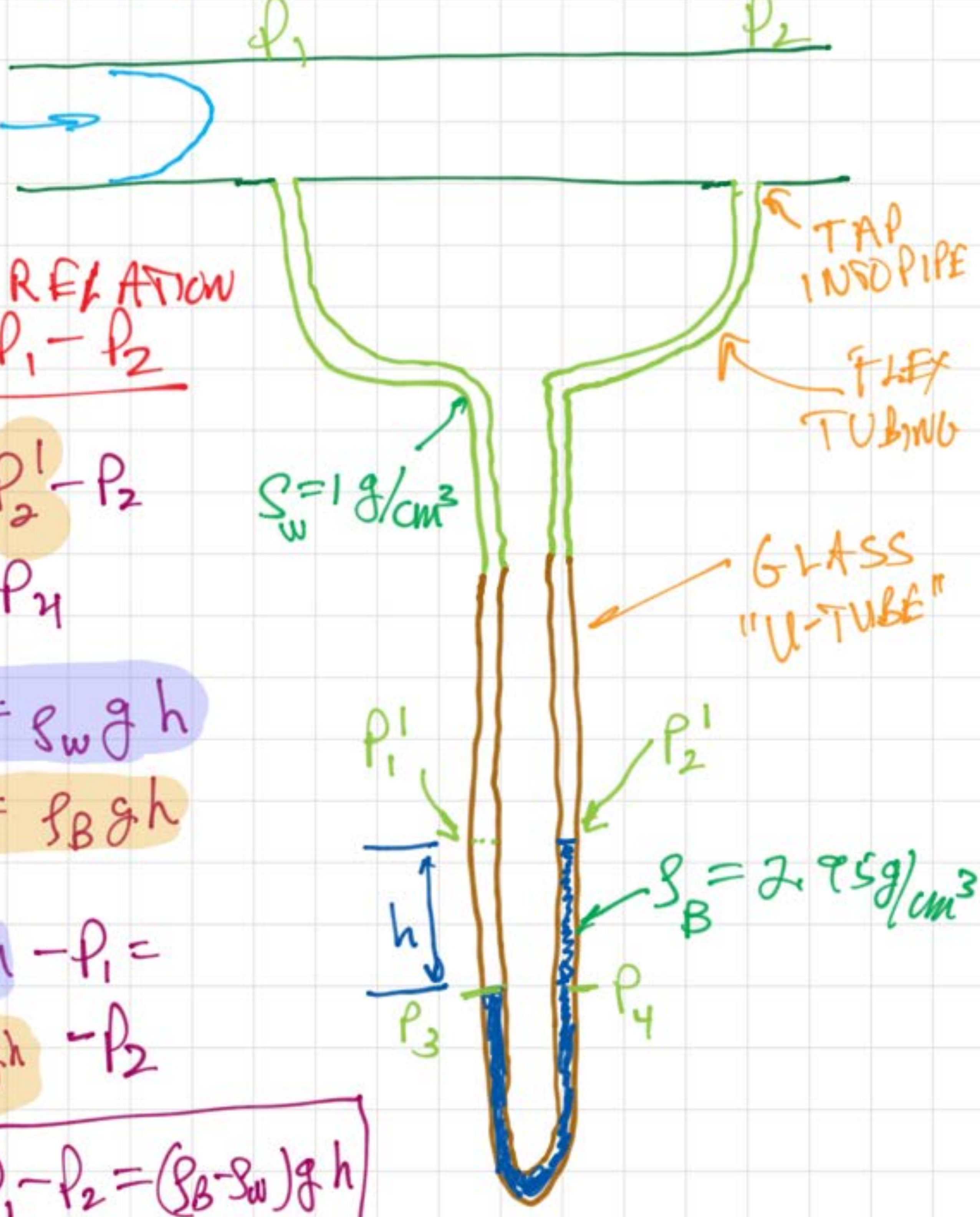
$\therefore \quad P = P_0 + \rho g z$

Z INCREASES DOWNWARD

P IS LINEAR IN Z.

BASIC EQUATION OF HYDROSTATICS

EXAMPLE: MANOMETER



FIND A RELATION FOR $P_1 - P_2$

$$P_1' - P_1 = P_2' - P_2$$

$$P_3 = P_4$$

$$P_3 - P_1' = \rho_w g h$$

$$P_4 - P_2' = \rho_B g h$$

SUBS:

$$P_3 - \rho_w g h - P_1 =$$

$$P_4 - \rho_B g h - P_2$$

$$\therefore \quad \boxed{P_1 - P_2 = (\rho_B - \rho_w) g h}$$

NOW CONSIDER CASE 1:

FLOW BY GRAVITY

$$0 = \frac{\mu}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} + \rho g_z$$

$$\frac{\mu}{r}\frac{d}{dr} r \frac{dv_z}{dr} = -\rho g_z$$

$$\frac{d}{dr} r \frac{dv_z}{dr} = -\frac{\rho}{\mu} g_z r$$

$$\int d\left(r \frac{dv_z}{dr}\right) = -\frac{\rho}{\mu} g_z \int r\,dr$$

$$r \frac{dv_z}{dr} = -\frac{\rho}{\mu} g_z \frac{r^2}{2} + C_1$$

$$\int dv_z = \int \left(-\frac{\rho}{\mu} g_z \frac{r}{2} + \frac{C_1}{r}\right) dr$$

$$v_z = -\frac{\rho}{\mu} g_z \frac{r^2}{4} + C_1 \ln r + C_2$$

BUT $C_1 = 0$

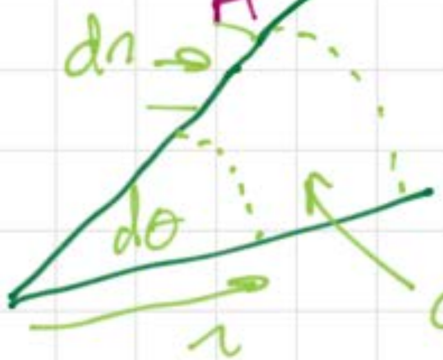
AT $v_z(R) = 0$

$$v_z(r) = \frac{\rho g_z}{4\mu}\left(R^2 - r^2\right)$$ PARABOLIC

WE WOULD LIKE AVERAGE $\langle v_z \rangle$

NEED VOLUMETRIC FLOW

$$Q = \int_A v_z(r,\theta)\, dA$$



$dA = (r\,d\theta)(dr)$

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z(r,\theta)\, r\, dr\, d\theta}{\int_0^{2\pi}\int_0^R r\, dr\, d\theta}$$

$$\langle v_z \rangle = \frac{\rho g_z R^2}{8\mu} = \tfrac{1}{2} v_{MAX}$$

$$Q = \langle v_z \rangle \pi R^2$$

$$= \frac{2\pi \, \rho g_z R^4}{8\mu}$$

VERY SENSITIVE TO THIS VALUE

Boundary conditions:

1. Fluid sticks to solid surfaces $V_x(0) = 0$
2. Fluid sticks to another immiscible fluid $V_x^{I}(h) = V_x^{II}(h)$
3. The shear stress is continuous across the fluid-fluid interface

$$\tau_{yx}^{I}(h) = \tau_{yx}^{II}(h)$$

$$\mu^{I} \frac{\partial V_x^{I}}{\partial y}\bigg|_h = \mu^{II} \frac{\partial V_x^{II}}{\partial y}$$

4. For liquid flowing next to quiescent air, the stress can be considered approximately 0.

$$\tau_{yx}(h) \approx 0$$

$$\frac{\partial V_x}{\partial y}(h) \approx 0$$

"FALLING FILM"