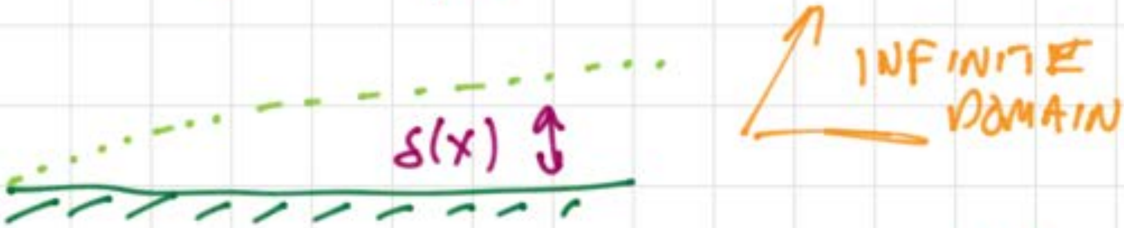
CBE 30357

12/5/17

1) QUICK RECAP OF BOUNDARY-LAYERS

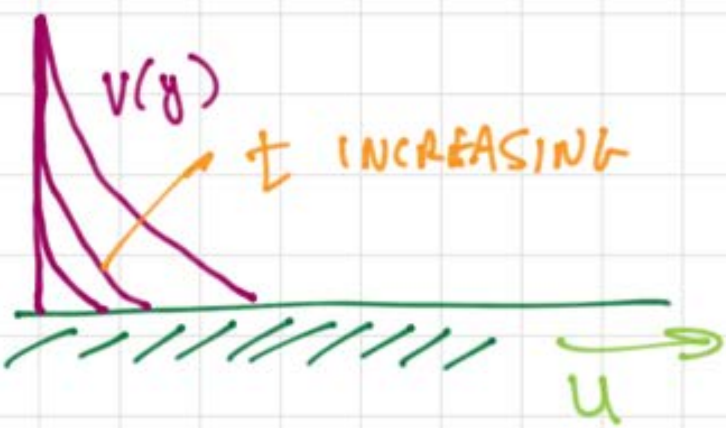
- INERTIA ~ VISCOUS TERMS
- $\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$



BOUNDARY-LAYER DEVELOPS W/ DISTANCE

2) STARTUP FLOWS, $U(t=0) = 0$

FLOW FIELD DEVELOPS IN TIME



X- N.S EQUATION:

$$\rho\left(U_0 V_x^* \frac{U_0}{L}\frac{\partial V_x^*}{\partial x^*} + V\, V_y^* \frac{U_0}{\delta}\frac{\partial V_x^*}{\partial y^*}\right) =$$

$$-\frac{\rho U_0^2}{L}\frac{\partial P^*}{\partial x^*} + \mu\left(\frac{U_0}{L^2}\frac{\partial^2 V_x^*}{\partial x^{*2}} + \frac{U_0}{\delta^2}\frac{\partial^2 V_x^*}{\partial y^{*2}}\right)$$

$$\ll\ll$$

CLEAR VARIABLES TO LEFT

$$\frac{\rho\delta^2 U_0}{\mu L}\left(V_x^*\frac{\partial V_x^*}{\partial x} + V_y^*\frac{\partial V_x^*}{\partial y}\right) = -\frac{\partial P^*}{\partial x^*} + \frac{\partial^2 V_x^*}{\partial y^{*2}}$$

FOR OUR ANALYSIS TO WORK WE REQUIRE:

$$\frac{\rho\delta^2 U_0}{\mu L} = O(1)$$

THUS: $\delta^2 = \dfrac{\mu L}{\rho U_0}$

$Re = \dfrac{L U_0 \rho}{\mu}$

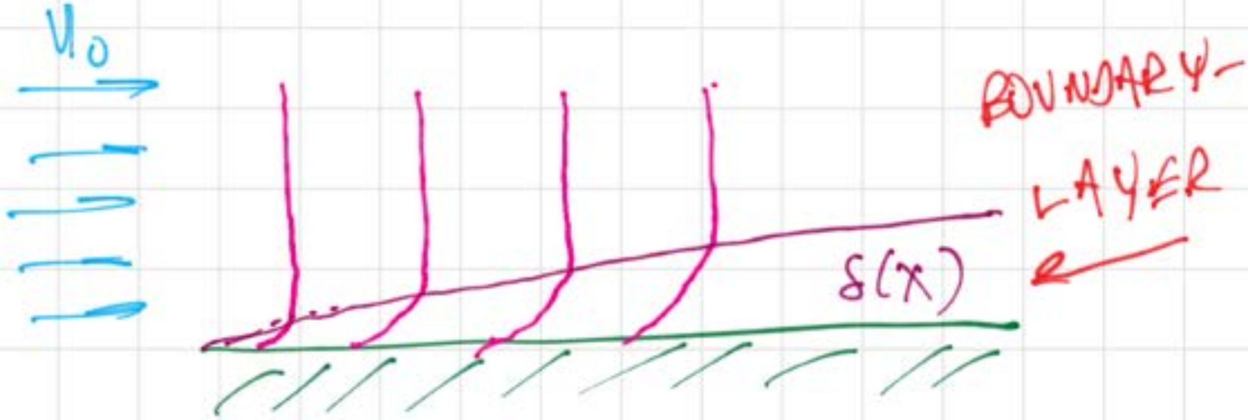
KEY RESULT!!

$$\boxed{\frac{\delta}{L} = \frac{1}{\sqrt{Re}}}$$

BOUNDARY-LAYER THICKNESS

RESULT FROM EQUATION SCALING

WE WANT TO ANALYZE:



BOUNDARY-LAYER EQUATIONS

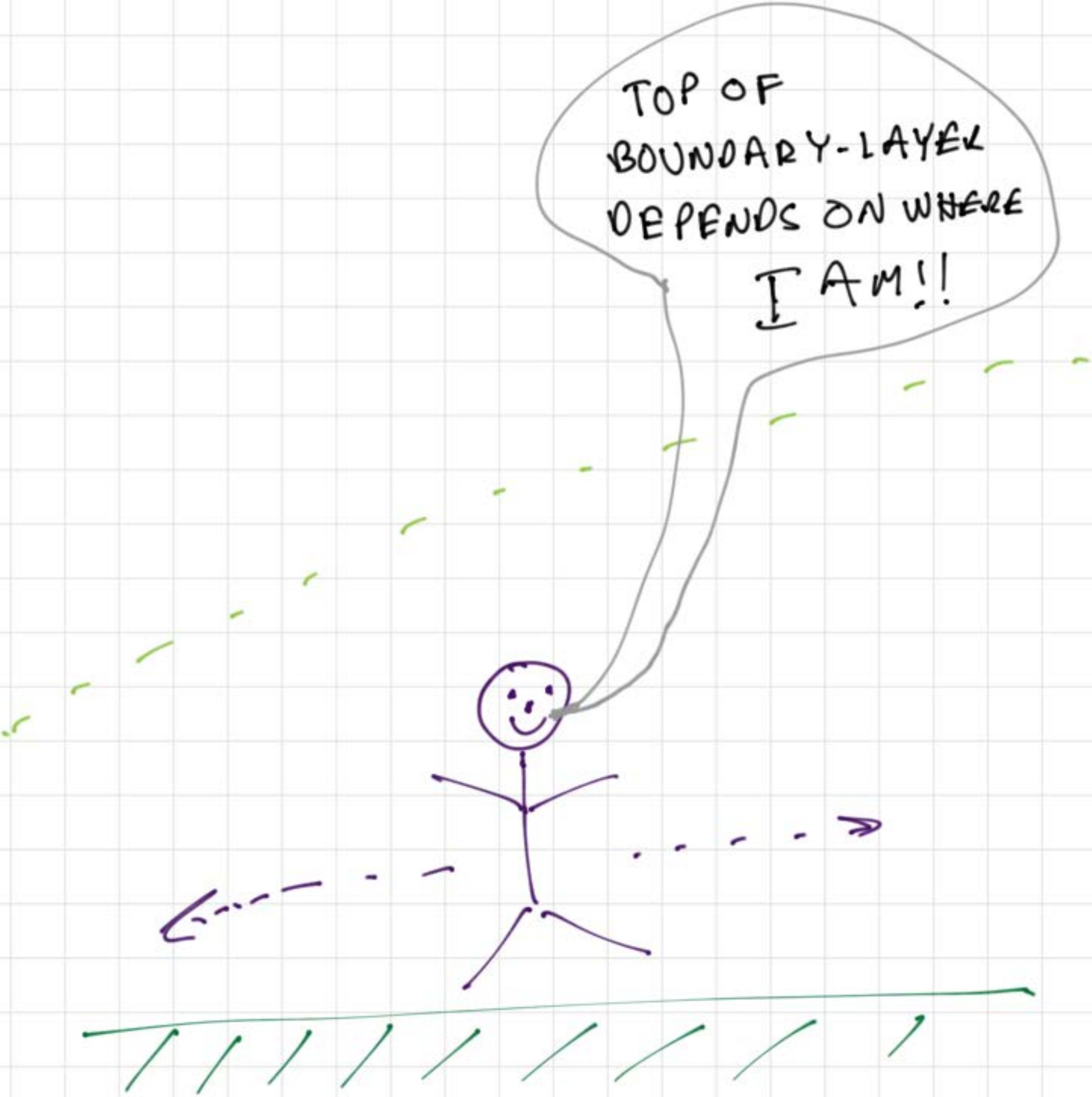
$$\boxed{\begin{gathered}\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \\ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2} \\ \frac{\partial P}{\partial y} = 0\end{gathered}}$$

CONSIDER "FLAT PLATE"

$$\Delta\left(\frac{u^2}{2} + gh + \frac{P}{\rho}\right) = 0$$

$$u_1 = u_2 \quad P_1 = P_2 \quad \therefore \quad \frac{\partial P}{\partial x} = 0$$

STILL HAVE COUPLED, NONLINEAR PDE'S ... CAN WE SIMPLIFY?


TOP OF BOUNDARY-LAYER DEPENDS ON WHERE I AM!!

IF THE PROFILES ARE SELF-SIMILAR THEN AS WE PROGRESS IN DISTANCE THERE IS A RELATIONSHIP BETWEEN x + y AS:

η => SIMILARITY VARIABLE

$$\eta \equiv \frac{y}{\sqrt{\nu x / u_0}}$$

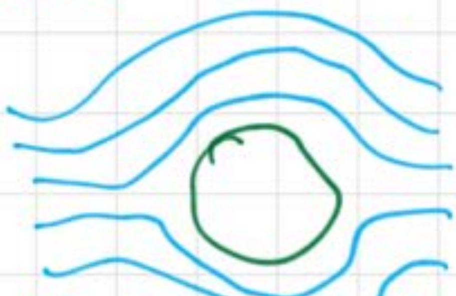
ONLY LENGTH SCALE TO MAKE →

← APPARENT CHANGE IN δ W/ DISTANCE

WE WILL ALSO USE THE "STREAM FUNCTION", ψ

DEFINED FROM CONTINUITY

$$\left\{ v_x \equiv \frac{\partial \psi}{\partial y}, \quad v_y \equiv -\frac{\partial \psi}{\partial x} \right.$$



LINES OF ψ = CONSTANT

$$\psi(x,y) \equiv f(\eta)\sqrt{\nu x u_0}$$

FROM THIS:

$$V_x = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$V_y = -\frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \ldots$$

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0$$

BC'S

$f(\eta) = 0 \quad V_y(y=0, x) = 0$

$f'(\eta) = 0 \quad V_x(y=0, x) = 0$

$f'(\infty) = 1 \quad \begin{cases} V_x(x=0, y) = U_0 \\ V_x(x, y=\infty) = U_0 \end{cases}$

SOLVE BY SHOOTING METHOD

BOUNDARY-LAYER THICKNESS IS ABOUT $\eta = 5$

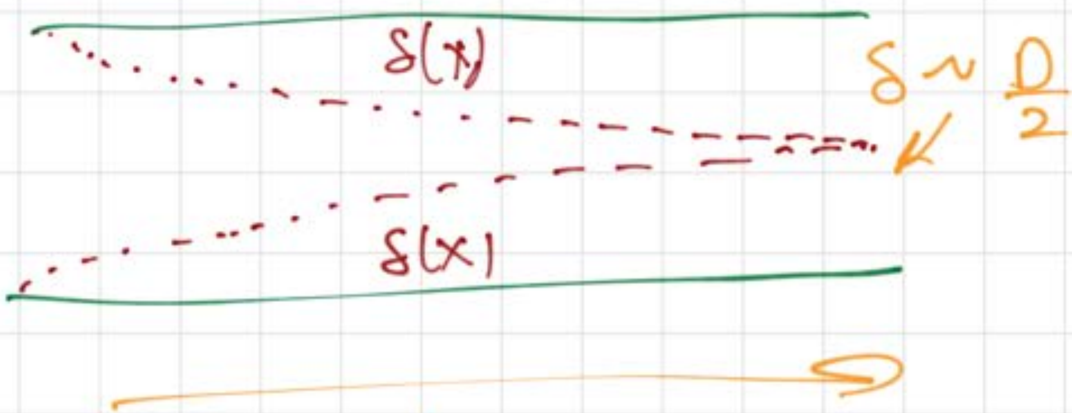
SO, WHAT IS y VALUE OF δ ?

$$\frac{y}{\sqrt{\frac{x\nu}{U_0}}} = 5 \quad \therefore \quad \delta = 5\sqrt{\frac{x\nu}{U_0}}$$

$$\delta = 5x\sqrt{\frac{\nu}{xU_0}}$$

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \qquad Re_x \equiv \frac{xU_0}{\nu}$$

LET US EXTEND THE DEVELOPING BOUNDARY-LAYER ON A PLATE TO DEVELOPING LAMINAR FLOW IN A PIPE (OR CHANNEL)



ENTRANCE LENGTH: L_e

$$\frac{D}{2} = \delta = 5x\sqrt{\frac{\nu}{x u_0}} \qquad x \to L_e$$

$$\frac{D^2}{4} = 25\,\frac{\nu L_e}{u_0}$$

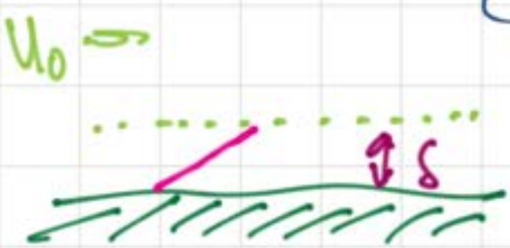
$$\frac{L_e}{D} \sim \frac{1}{100}\,\frac{u_0 D}{\nu} = 0.01\,Re$$

WHAT IS WALL SHEAR STRESS?

$$\tau_w = \mu \frac{\partial V_x}{\partial y} = \mu U_0 \left(\frac{U_0}{x\nu} \right)^{1/2} f''(0)$$

$$\tau_w = .332 \mu U_0 \left(\frac{U_0}{\nu x} \right)^{1/2}$$

COMPARE TO:



$$\tau_w = \mu \frac{U_0}{\delta} \quad \ldots$$ "LINEAR PROFILE APPROXIMATION"

$$\tau_w = \mu \frac{U_0}{5x \, Re_x^{-1/2}}$$

$$= .2 \mu U_0 \left(\frac{U_0}{\nu x} \right)^{1/2}$$

SO, .2 COMPARED TO .33 IS PRETTY CLOSE!!

WHAT IS EFFECT OF PRESSURE GRADIENT ON BOUNDARY-LAYER?

BERNOULLI EQ.

$$\Delta\left(\frac{V^2}{2}+gh+\frac{p}{\rho}\right)=0$$

$$\mathcal{P}=p+\rho g h$$

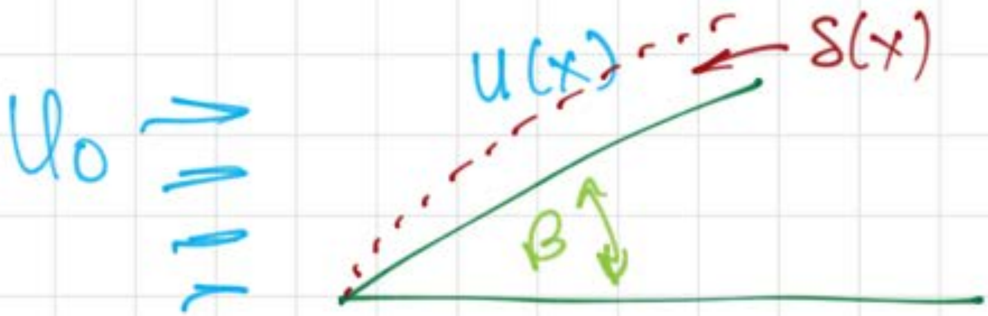
THEN DIFFERENTIATE:

$$\underbrace{V\frac{dV}{dx}}_{\text{CHANGE IN FARAWAY FLOW}}+\frac{1}{\rho}\frac{d\mathcal{P}}{dx}=0$$

REPLACE $\frac{d\mathcal{P}}{dx}$ IN BOUNDARY-LAYER EQ

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=\underbrace{U\frac{dU}{dx}}+\nu\frac{\partial^2 v_x}{\partial y^2}$$

FLOW PAST A WEDGE



$$U(x) = A x^m$$

$$m \equiv \frac{\beta}{(2\pi - \beta)}$$

INCLUDE THIS NEW TERM IN ANALYSIS:

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0$$

SYSTEM OF ODE'S:

$$y_1' = y_2$$

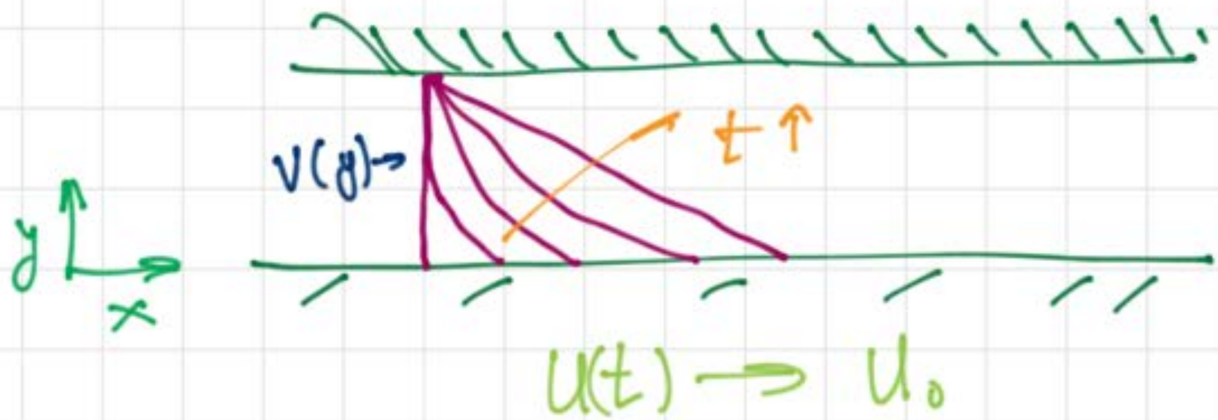
$$y_2' = y_3$$

$$y_3' = -\frac{m+1}{2} y_1 y_3 + m(1 - y_2^2)$$

SOLVE ....

SEE "BOUNDARYLAYER..2017"

PARALLEL PLATE GEOMETRY
STARTUP FLOW BY MOVING BOTTOM PLATE



WHAT IS RELEVANT EQ ?

$V_x(y) = \;\therefore\; \cancel{f(x)}$ (UNIFORM IN X)

FROM CONTINUITY $\frac{\partial V_x}{\partial x} = 0 \;\therefore\; \frac{\partial V_y}{\partial y} = 0 \;\therefore\; V_y = 0$

$$\frac{\partial v_x}{\partial t} + \cancelto{0}{v_x \frac{\partial v_x}{\partial x}} + \cancelto{0}{v_y \frac{\partial v_x}{\partial y}} = \cancelto{0}{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$ "HEAT EQUATION"

SEPARATION OF VARIABLES:

IF $V_x = Y(y)\,T(t)$ SUBS

$$\frac{T'(t)}{T(t)} = \nu \frac{Y''(y)}{Y(y)} = -\lambda^2$$

$T' = -\lambda^2 T \rightarrow$ EXPONENTIAL DECAY

$Y'' + \lambda^2 Y = 0 \rightarrow$ SINES & COSINES

EIGENVALUE

SOLUTION REQUIRE INFINITE SUM OF $\sin\left(\frac{\pi y j}{b}\right)$

YOU WILL PROBABLY GET A CHANCE TO SOLVE THIS NEXT SEMESTER..

PERHAPS ON AN EXAM...

- ANSWER IS READILY AVAILABLE

```
heqn = D[u[y, t], t] == ν D[u[y, t], {y, 2}];
```

The initial condition is, ux= 0

```
ic = u[y, 0] == 0;
```

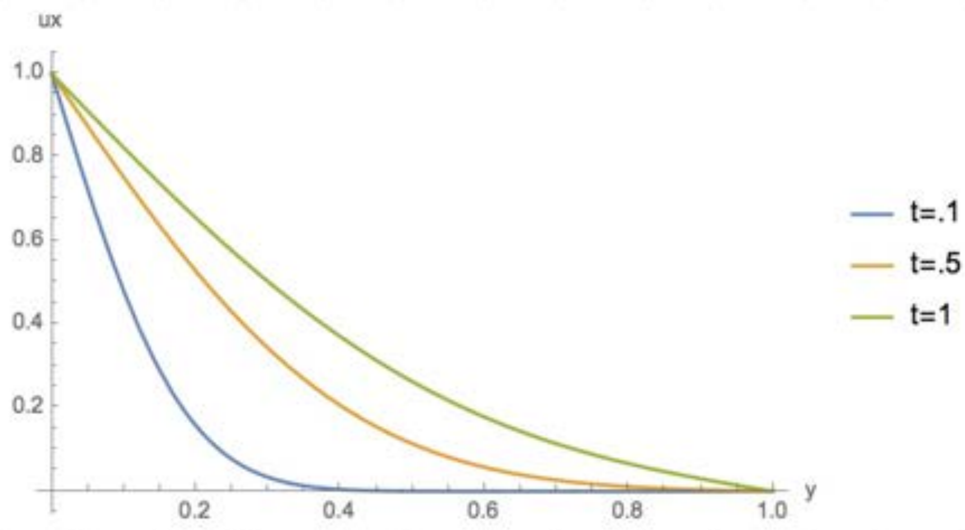
The boundary conditions for a channel of height b are:

```
bc = {u[0, t] == u0, u[b, t] == 0};
```

For these boundary conditions, the problem is easily solved

```
sol = DSolve[{heqn, ic, bc}, u[y, t], {y, t}]
```

$$\left\{\left\{u[y,\,t] \to u0 - \frac{u0\,y}{b} - \frac{2\sum_{K[1]=1}^{\infty} \frac{e^{-\frac{\pi^2\,t\,\nu\,K[1]^2}{b^2}}\,u0\,\mathrm{Sin}\left[\frac{\pi\,y\,K[1]}{b}\right]}{K[1]}}{\pi}\right\}\right\}$$



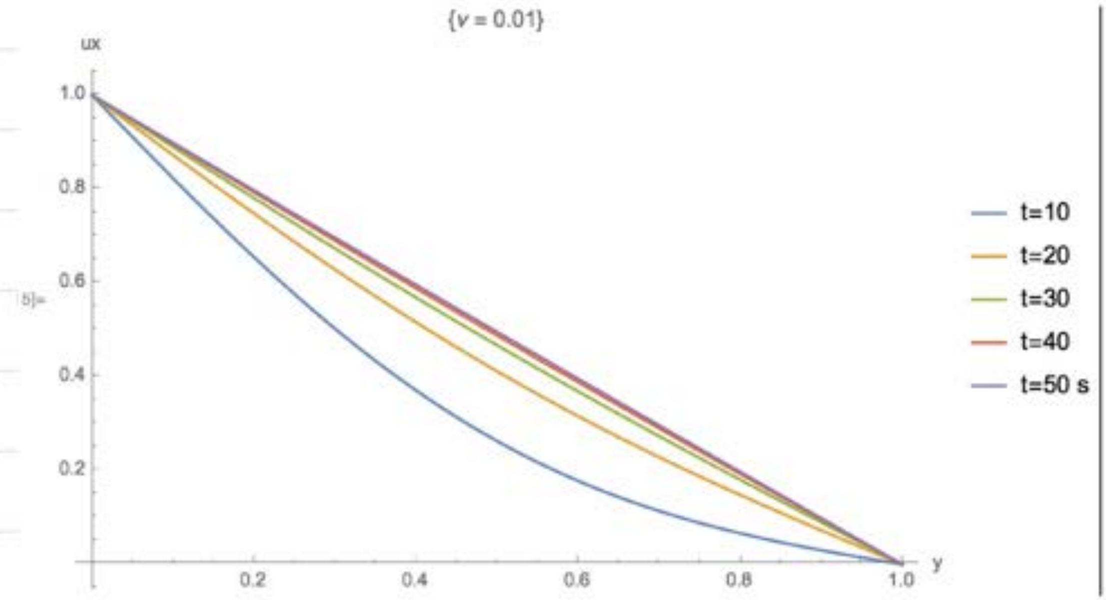
HOW LONG DOES
START UP LAST?

FROM SCALING...:

$$\tau \sim \frac{b^2}{\nu} = \frac{(1\ cm)^2}{(.01\ cm^2/s)}$$

$$\sim 100\ s$$

WE CAN CHECK...



$\sim$ 50 IS ENOUGH

$$\exp\left(-\frac{\pi^2}{b^2}\nu t\right)$$

NEED ARGUMENT TO BE ≥ 4

$$t = \frac{4b^2}{\pi^2 \nu} = \underline{\underline{40}}\ s$$

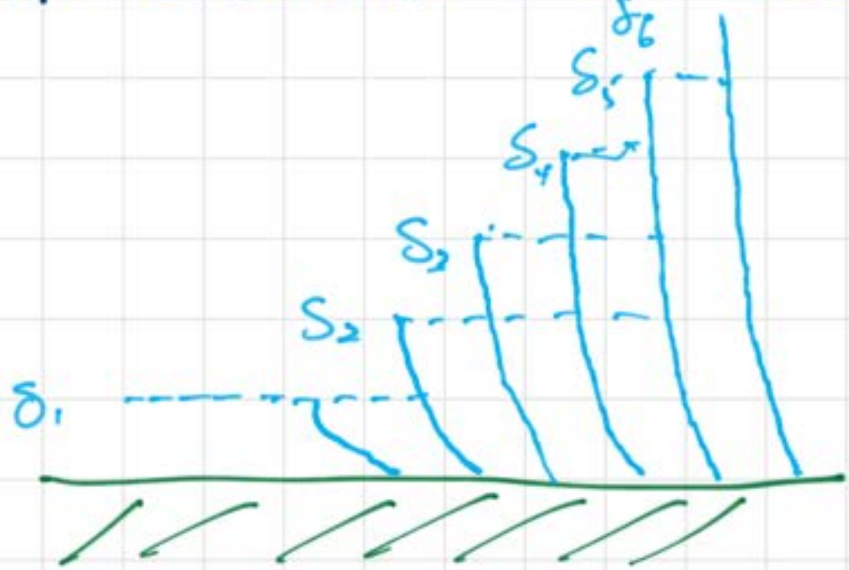
AS $b \uparrow$ OR $t \downarrow$ THIS SOLUTION WILL REQUIRE MANY MORE MODES TO CONVERGE.

IN THE LIMIT $b \to \infty$, ANOTHER APPROACH IS NEEDED

(USE LAPLACE TRANSFORM)

AGAIN: NO GEOMETRIC LENGTH SCALE

ON A INFINITE DOM..., WE EXPECT SELF-SIMILAR PROFILES



$$\delta \sim \sqrt{\nu t}$$

$$\frac{\partial V_x}{\partial t} = \nu \frac{\partial^2 V_x}{\partial y^2}$$

$$V_x(y, t<0) = 0$$

$$V_x(0, t\geq 0) = U_0$$

$$V_x(\infty, t\geq 0) = 0$$

SAME

HYPOTHESIZE: BECAUSE OF SELF SIMILARITY:

$$\frac{V_x}{U_0} \equiv f(\eta)$$

SIMILARITY VARIABLE

$$\eta \equiv \frac{y}{\sqrt{4\nu t}}$$

TRANSFORM EQUATION USING CHAIN RULE

(SEE MMA NOTEBOOK)

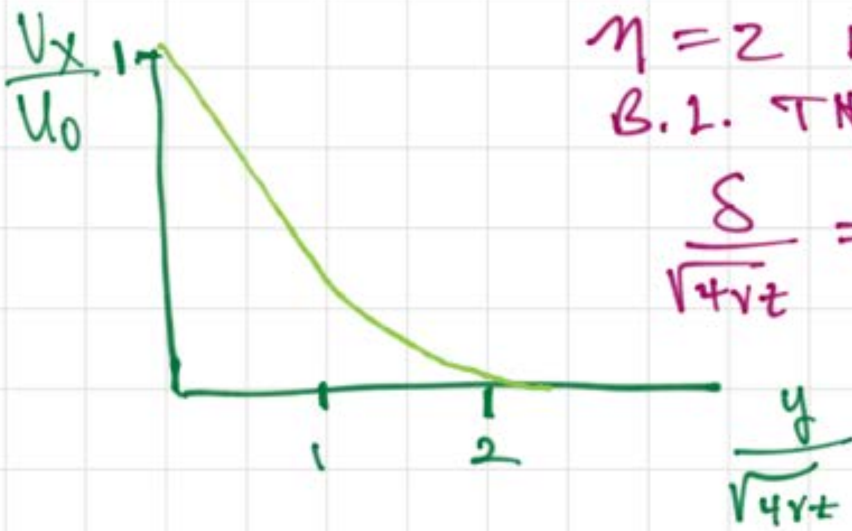
$$f'' + 2\eta f' = 0$$

$$f(0) = 1$$

$$f(\infty) = 0 \quad \text{(2 B.C.'S COLLAPSE)}$$

YOU COULD SOLVE BY INTEGRATING TWICE

$$f(\eta) = 1 - \mathrm{erf}(\eta)$$



$\eta = 2$ EFFECTIVE B.L. THICKNESS

$$\frac{\delta}{\sqrt{4\nu t}} = 2 \Rightarrow$$

$$\delta(t) = 4\sqrt{\nu t}$$