CBE 30357

10/10/17

TOPICS

1) REVIEW: USING PRINCIPLES OF OPTIMIZATION TO DETERMINE OPTIMAL SIZE OF A BLOOD VESSEL AND DIAMETER RATIO ACROSS A BRANCH

- TOTAL COST = CAPITAL COST + OPERATING COST

- $R_1^3 = 2 R_2^3$

2) HOW TO CALCULATE FLOWRATE PRESSURE DROP BEHAVIOR FOR NON-CIRCULAR CONDUITS...

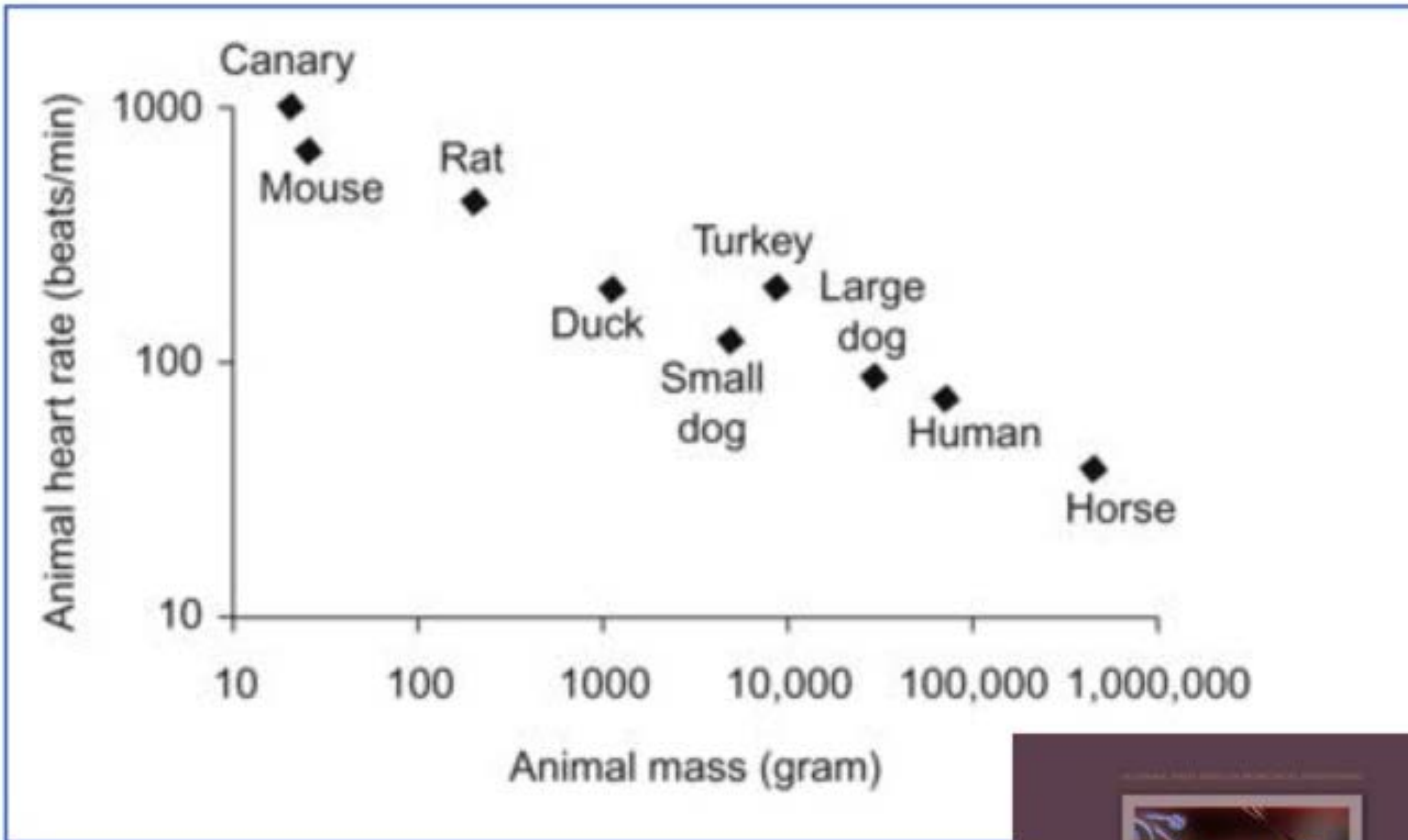
- SOLVE PDE
- USE "HYDRAULIC DIAMETER"

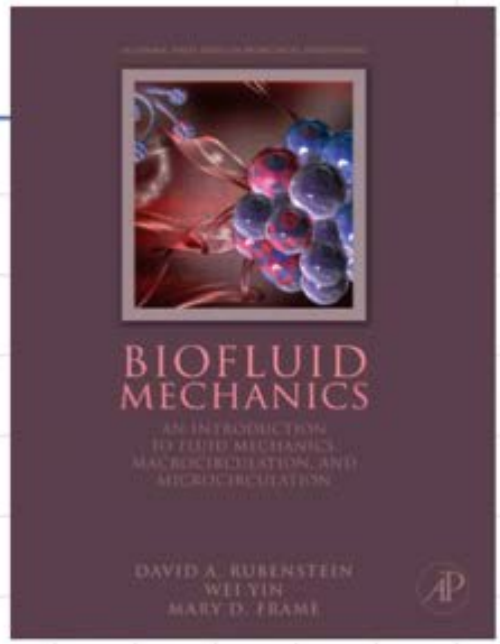
3) NONDIMENSIONALIZATION OF NAVIER-STOKES EQUATIONS

WE CAN ACCESS USEFUL ANALYTICAL SOLUTIONS IN CERTAIN PARAMETER RANGES

RECALL:

THERE MUST BE A REASON FOR THIS....





"ALLOMETRIC" DATA

BLOOD PRESSURE v. MASS

NOT REALLY CONSTANT!!

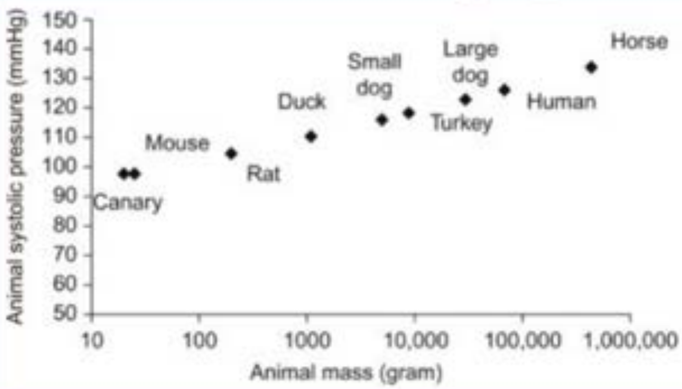
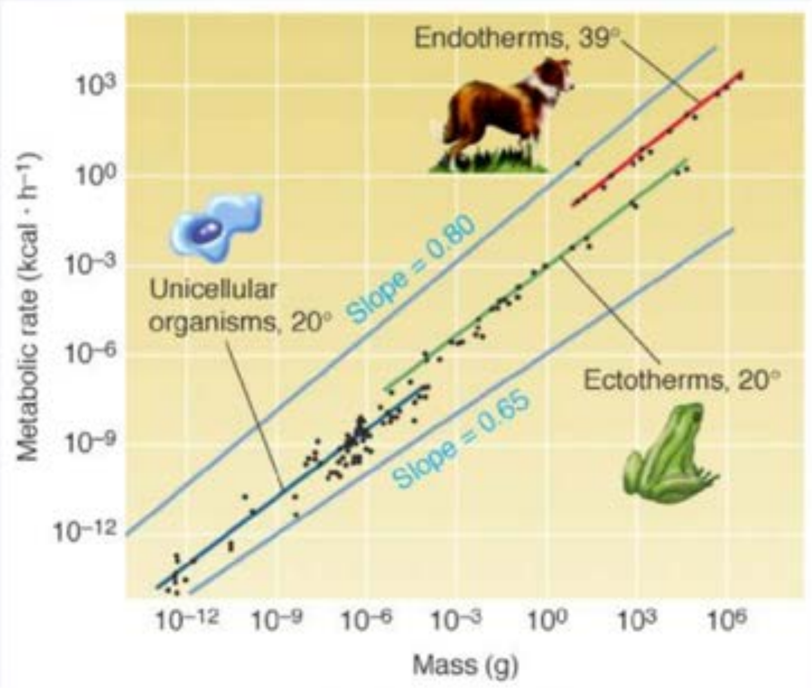


Figure 5.9 The relationship between animal mass and mean systolic pressure. As the mass of the animal increases, there is a general increase in systolic pressure. The relationship between these two measurements can be correlated to many different properties of the animal, as described in the text.

METABOLIC POWER(KLEIBER'S LAW)

$P \sim M^{.75}$



HEART RATE — MASS

- rho $f^3 V_h^{5/3} \sim M^{.75}$
- Further $V_h \sim M$
  - http://www.biologyreference.com/Re-Se/Scaling.html
- Which gives…
  - $f \sim M^{-.31}$
- Interesting… I don't know how "correct" it is
- There are other allometric observations….

WHY DO BLOOD VESSELS BRANCH LIKE THIS?

TABLE II

VESSELS IN TABLE I GROUPED ACCORDING TO RANK

Vessel rank	Σr^2	Σr^3	Σr^4
	mm^2	mm^3	mm^4
0	2.2	3.4	5.1
1	3.8	1.9	0.94
2	4.0	1.2	0.36
3	8.6	0.54	0.043
4	20	0.51	0.013
5	140	1.5	0.021
6	200	1.8	0.019
7	650	2.4	0.0095
6′	380	5.5	0.10
5′	200	7.6	0.30
4′	120	6.3	0.37
3′	39	5.8	1.1
2′	25	19	14
1′	22	26	31
0′	9	27	81

The vessels of Table I have been grouped according to rank and the sums of r^2, r^3, and r^4 have been calculated for each rank.

The answer has to be that "nature" (millions of years of evolution) has performed optimization, to maximize fitness of organism, as constrained by physical laws.

'RECALL': MURRAY'S LAW

COST = COST OF PIPE + COST TO PUMP

TOTAL METABOLIC POWER REQUIRED FOR CIRCULATION SYSTEM = POWER TO MAINTAIN BLOOD + VESSELS + POWER NEEDED BY HEART TO PUMP

TOTAL POWER $= \beta \pi R^2 L + \Delta P Q$

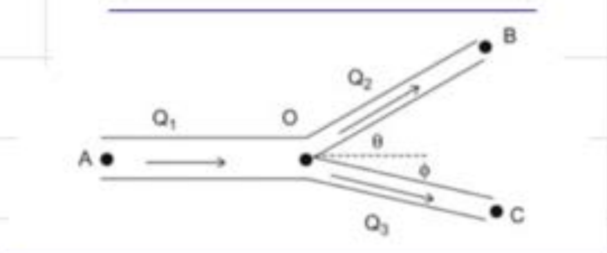


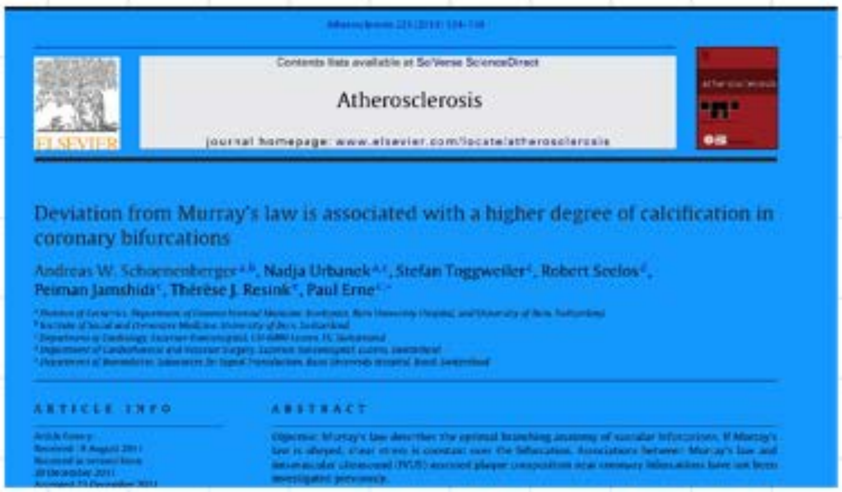
Figure 5.14 Flow rate at a bifurcation

$$r_1^3 = r_2^3 + r_3^3 \tag{5.26}$$

$$\cos\theta = \frac{r_1^4 + r_2^4 - r_3^4}{2r_1^2 r_2^2}$$

$$\cos\phi = \frac{r_1^4 - r_2^4 + r_3^4}{2r_1^2 r_3^2}$$

$$\cos(\theta + \phi) = \frac{r_1^4 - r_2^4 - r_3^4}{2r_2^2 r_3^2}$$



Contents lists available at SciVerse ScienceDirect

Atherosclerosis

journal homepage: www.elsevier.com/locate/atherosclerosis

Deviation from Murray's law is associated with a higher degree of calcification in coronary bifurcations

Andreas W. Schoenenberger, Nadja Urbanek, Stefan Toggweiler, Robert Seelos, Peiman Jamshidi, Thérèse J. Resink, Paul Erne

ARTICLE INFO

ABSTRACT

STILL MENTIONED IN LITERATURE

FOR AN EVEN SPLIT $\theta = \phi$

$\cos\theta = \frac{1}{\sqrt[3]{2}} \qquad \theta \approx \frac{\pi}{4.8}$

For a pressure driven flow, we could get an exact solution for an infinitely wide channel and a circular pipe.

This is nothing short of "great" --except if the channel is square!

So, what do we do?

IF $h/w < \sim .1$ WE ARE PROBABLY OK... ("HOW OK"?)

IF $h/w \approx 1$ WE EXPECT THAT INFINITELY WIDE CHANNEL IS NOT ACCURATE...

RECTANGULAR CHANNEL

CROSSSECTION : h x w

x direction

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x\frac{\partial v_x}{\partial x}} + \cancel{v_y}\frac{\partial v_x}{\partial y} + \cancel{v_z}\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \cancel{\rho g_x} \tag{3.3.26a}$$

y direction

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x\cancel{\frac{\partial v_y}{\partial x}} + \cancel{v_y\frac{\partial v_y}{\partial y}} + \cancel{v_z\frac{\partial v_y}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial y}} + \mu\left[\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho \cancel{g_y} \tag{3.3.26b}$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_x\cancel{\frac{\partial v_z}{\partial x}} + \cancel{v_y\frac{\partial v_z}{\partial y}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial z}} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \tag{3.3.26c}$$

Figure 3.7 Flow in a rectangular channel of height h and width w. The section is through the plane $z = 0$.

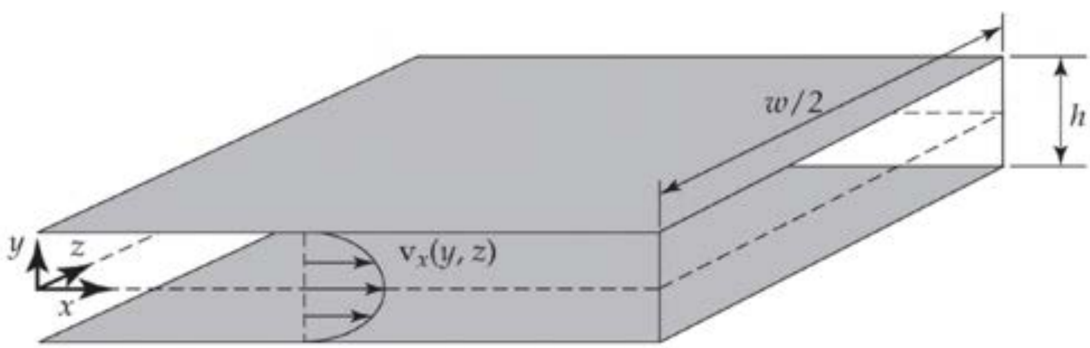
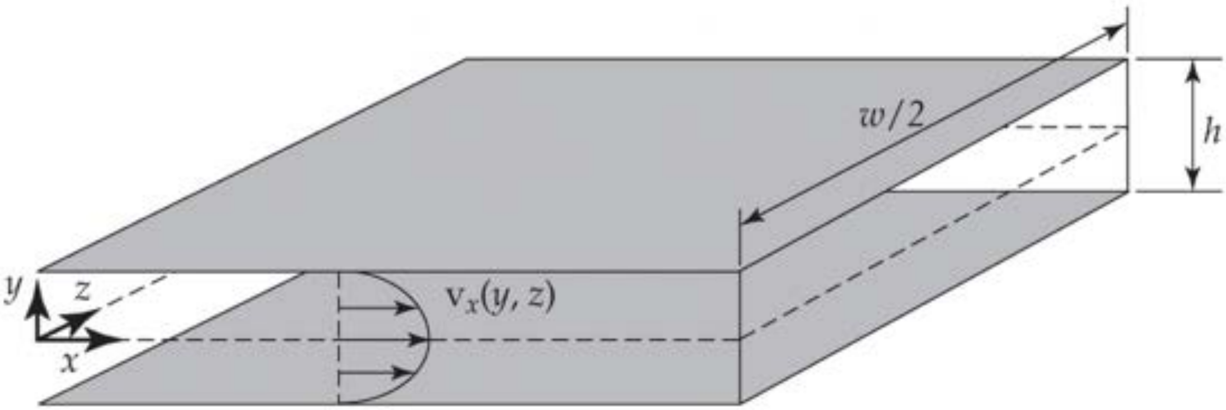


Figure 3.7 Flow in a rectangular channel of height h and width w. The section is through the plane $z = 0$.



FOR A STEADY, FULLY-DEVELOPED 2-D FLOW OF A NEWTONIAN FLUID . . . .

$$0 = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

POISSON'S EQ

SOLUTIONS ARE WELL KNOWN !!

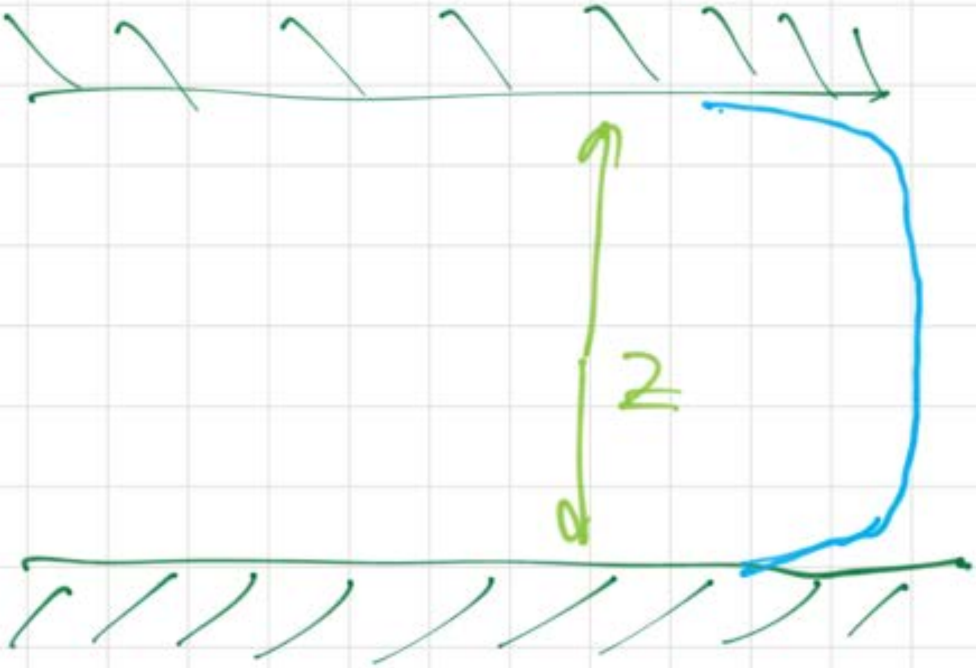
THIS EQUATION IS EASILY SOLVED BY SEPARATION OF VARIABLES

THE TEXT SHOWS THIS ON pp 138–139

EIGEN VALUE PROBLEM

and the solution for the velocity field is

$$\mathrm{v}_x(y, z) = \frac{\Delta p h^2}{8\mu L}\left(1 - \frac{4y^2}{h^2}\right) - \frac{\Delta p h^2}{8\mu L}\sum_{n=0}^{\infty}\frac{32(-1)^n \cosh\big((2n+1)\pi z/h\big)\cos\big((2n+1)\pi y/h\big)}{(2n+1)^3\pi^3\cosh\big((2n+1)\pi w/2h\big)}.$$



SIMILAR SHAPE TO $V_x(y)$.

PRESSURE DROP OR
SHEAR STRESS ...
INTEGRATE

$$\tau_{yx} = \frac{\mu}{W} \int_{-W/2}^{W/2} \left.\frac{\partial v_x}{\partial y}\right|_{y=-h/2} dz$$

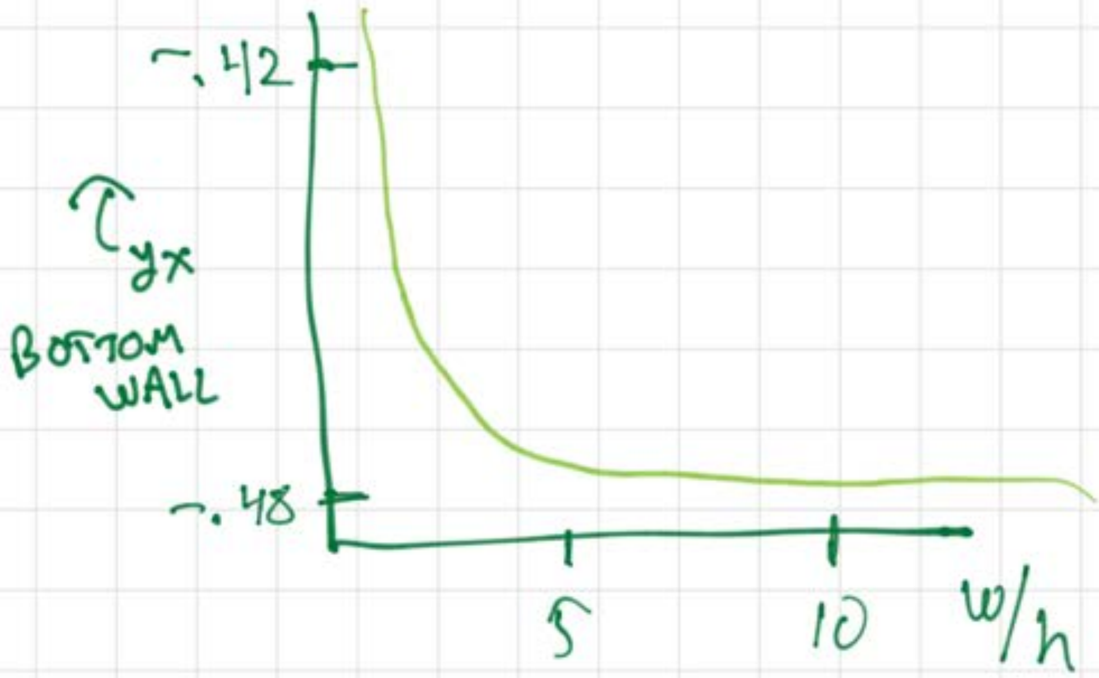
$$= \frac{\Delta P h}{2L}\left[1 - 16\frac{h}{W}\sum_{n=0}^{\infty}\frac{(-1)^n \tanh(2n+1)\pi\frac{W}{2h}}{(2n+1)^3\pi^3}\right]$$

ASPECT RATIO

$$\tau_{yx} \cong \left(\frac{-\partial P}{\partial x}\right) h\left(\frac{1}{2} - .25\,\varepsilon\right)$$

$$\varepsilon \equiv \frac{h}{W}$$

BOTTOM WALL, DOES NOT INCLUDE SIDES



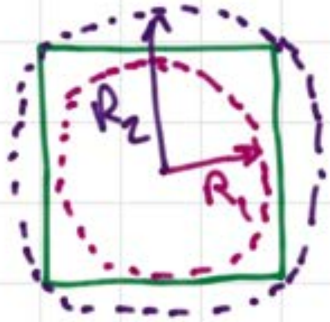
SHOULD WE TAKE TIME TO LEARN HOW TO DO ANALYICAL SOLUTION???

HOW ABOUT...

NOT THIS PDE.

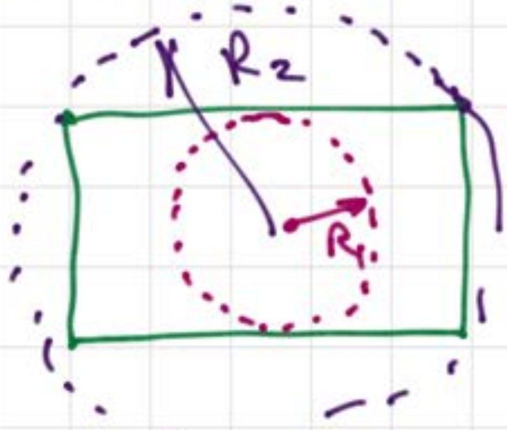
A NUMERICAL SOLUTION IS JUST A FEW LINES OF CODE... AND ANY SHAPE CAN BE DONE..

CAN WE "BRACKET" ANSWER



SOLVE FOR R_1 + R_2 AND EXPECT SQUARE TO BE BETWEEN THESE RESULTS ...

OK BUT:

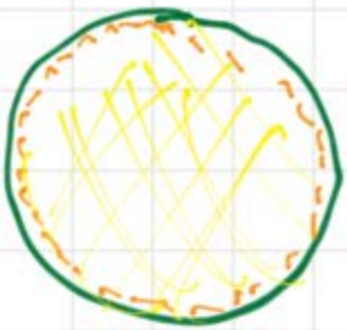


IS THERE A RATIONAL WAY TO PICK AN "R" FOR THE RECTANGLE?

IF SO, COULD USE WHAT WE ALREADY KNOW!!

"HYDRAULIC" DIAMETER

$$d_H \doteq \frac{4 \text{ CROSS SECTION AREA}}{\text{WETTED PERIMETER}}$$



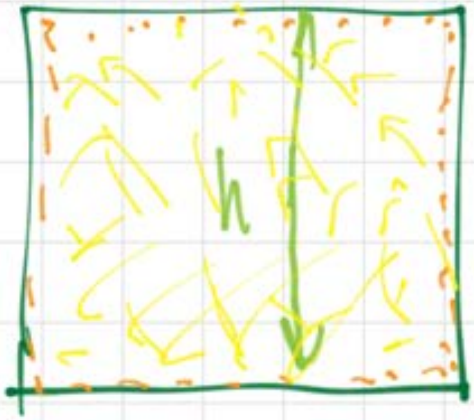
$2\pi \cdot R$

πR^2

$$d_H = \frac{4\pi R^2}{2\pi R} = 2R = D$$

SO THE "IDEA" WORKS FOR CIRCLE

SQUARE



$4h$

h^2

$$d_H = \frac{4h^2}{4h} = h$$

$$d_H = \frac{4wh}{2w + 2h}$$

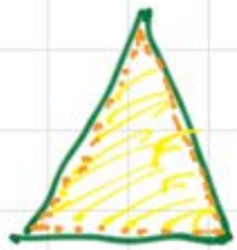
THIS IS CURIOUS

As $\frac{h}{w} \to 0 \quad d_H \to 2h$

HYDRAULIC DIAMETER

USE d_H FOR "DIAMETER" IN STANDARD CORRELATIONS

$$d_H = \frac{4\ \text{AREA}}{\text{PERIMETER}}$$



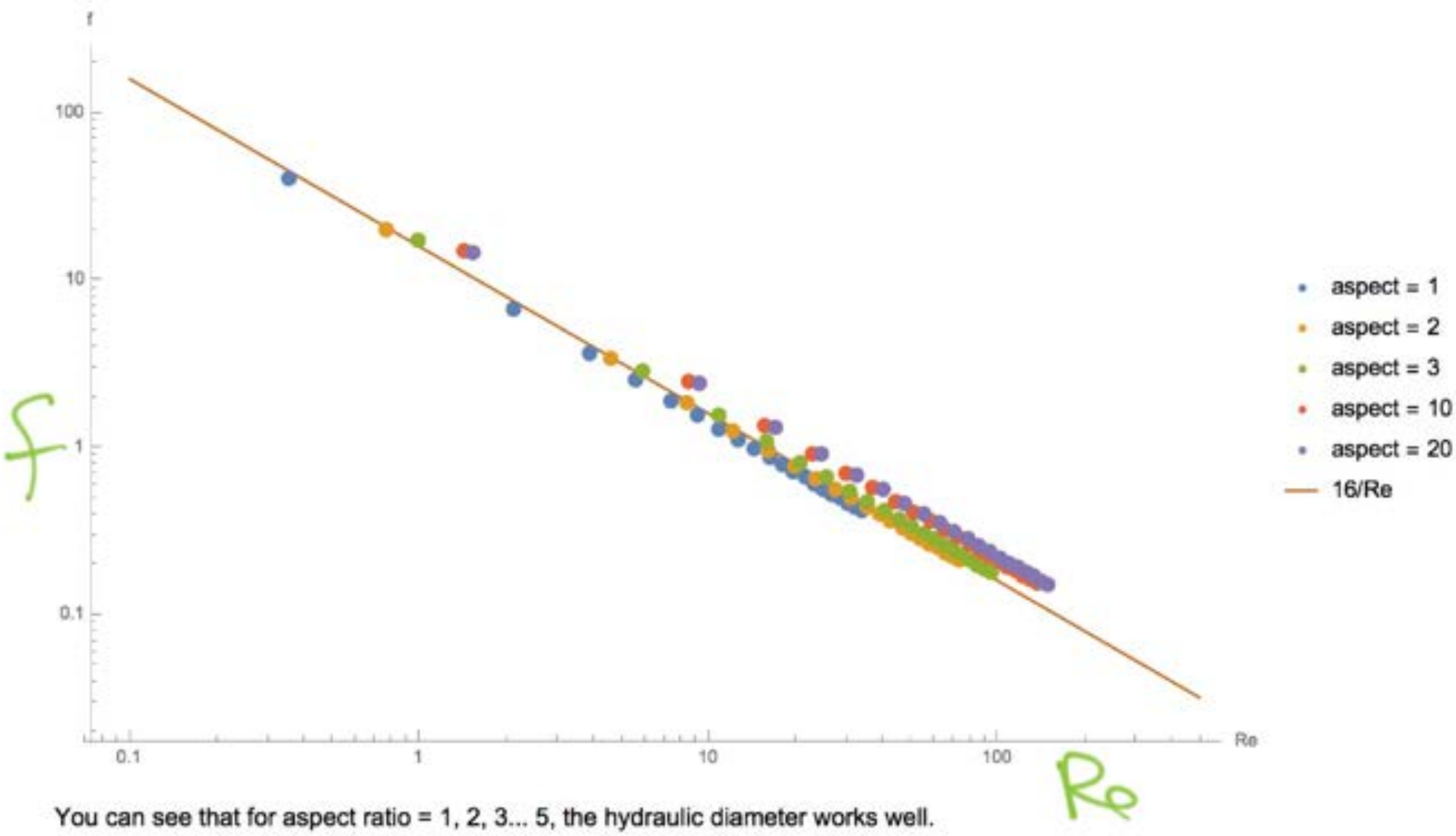
$$Re = \frac{\rho d_H V}{\mu} \qquad f = \frac{\Delta P\, d_H}{2 L \rho V^2}$$

$$f = \frac{16}{Re}, \quad f = 0.079\, Re^{-.25}$$

HOW WELL DOES IT WORK?

LAMINAR FLOW: RECTANGULAR CROSS-SECTION

```
ListLogLogPlot[{ff1, ff2, ff3, ff10, ff20, ff16Re},
 PlotLegends → {"aspect = 1", "aspect = 2", "aspect = 3", "aspect = 10", "aspect = 20", "16/Re"},
 Joined → {False, False, False, False, False, True}, AxesLabel → {"Re", "f"}]
```



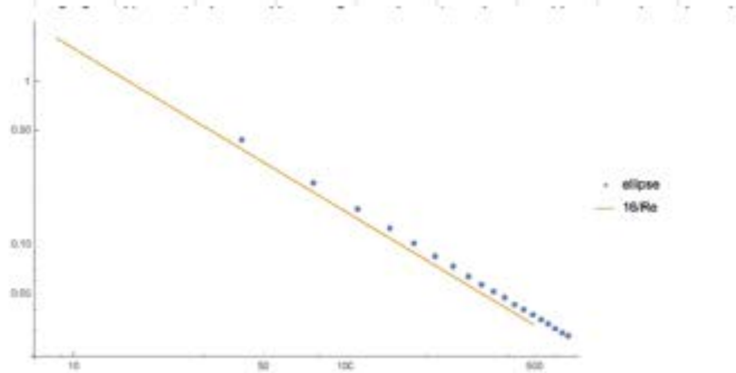
You can see that for aspect ratio = 1, 2, 3... 5, the hydraulic diameter works well.

SHOULD BE SIMILAR AGREEMENT FOR TURBULENT FLOW

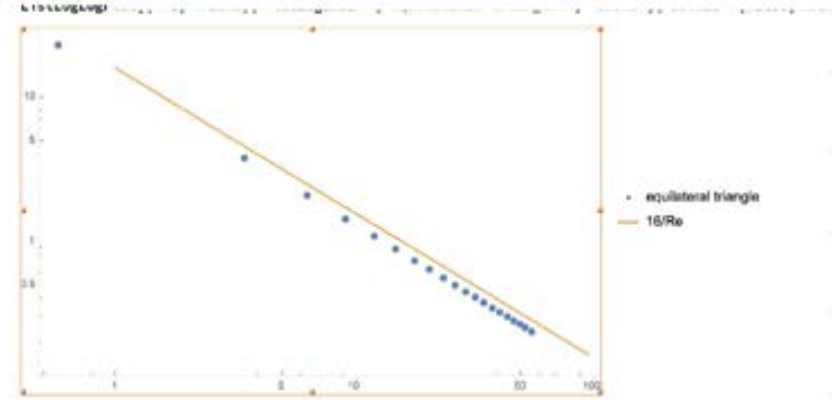
± 20% ACCURACY IS USUALLY OK FOR PIPE FLOW CALCULATIONS

MORE SHAPES

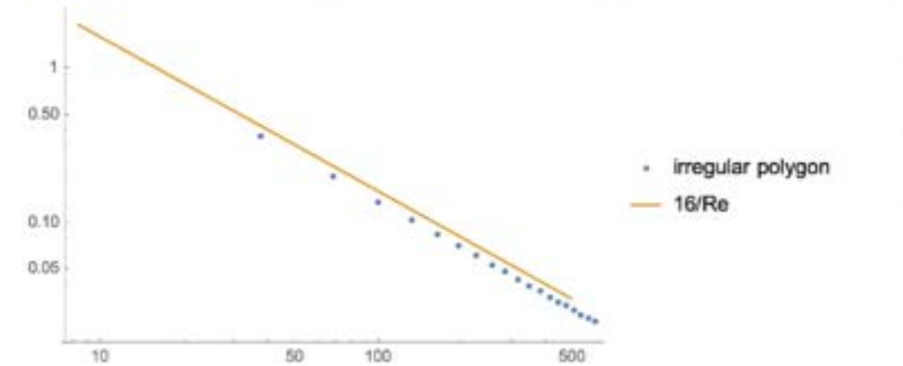
Graphics[Ω]



Graphics[Ω]



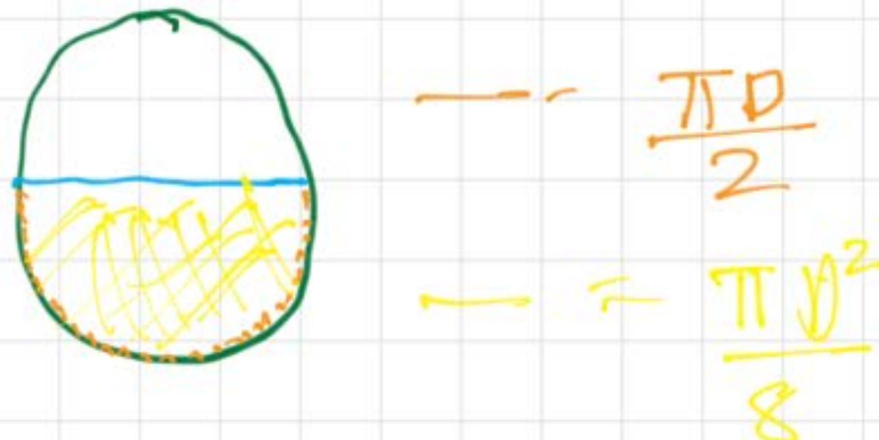
Graphics[Ω]



NOTE: "WETTED PERIMETER" COMES FROM A PIPE THAT IS NOT FULL; DRAINAGE CHANNEL

WHILE THE IDEAS ARE SAME CONSULT THE APPROPRIATE LITERATURE

THE REASON THAT WE TALK ABOUT "WETTED PERIMETER"



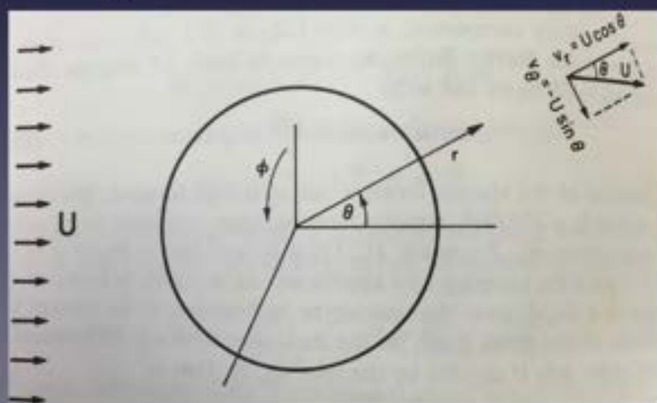
$$d_H = \frac{4\,\pi D^2/8}{\pi D/2} = D$$

THERE ARE OTHER CAVEATS FOR OPEN CHANNEL FLOWS...

NEXT MORE COMPLICATED FLOW . . .

Simple flow of interest

- Flow of liquid past a stationary sphere (which is the same as single particle moving (perhaps by gravity) through an otherwise quiescent liquid (or gas).
- Here is drawing from Denn's book:



Spherical coordinates (r, θ, ϕ) $$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}\right)$$

Spherical coordinates

r direction

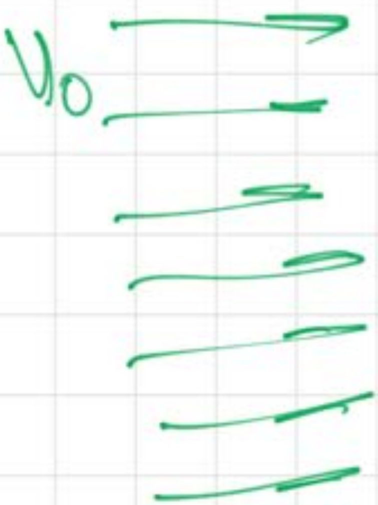
$$\rho\left[\cancel{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}} - \frac{v_\theta^2 + \cancel{v_\phi^2}}{r}\right] = -\frac{\partial p}{\partial r} + \cancel{\rho g_r}$$
$$+ \mu\left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}} - 2\frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta - \cancel{\frac{2}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}}\right] \quad (3.3.28a)$$

θ direction

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}} + \frac{v_\theta v_r}{r} - \cancel{\frac{v_\phi^2\cot\theta}{r}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \cancel{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}}\right] + \cancel{\rho g_\theta} \quad (3.3.28b)$$

ϕ direction

$$\rho\left(\cancel{\frac{\partial v_\phi}{\partial t}} + v_r\cancel{\frac{\partial v_\phi}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\phi}{\partial \theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}} + \cancel{\frac{v_\phi v_r}{r}} + \cancel{\frac{v_\theta v_\phi}{r}}\cot\theta\right) = -\cancel{\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cancel{\frac{\partial v_\phi}{\partial r}}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\cancel{\frac{\partial v_\phi}{\partial \theta}}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}} + \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}} - \cancel{\frac{v_\phi}{r^2\sin^2\theta}} + \cancel{\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}}\right] + \cancel{\rho g_\phi} \quad (3.3.28c)$$



$$\rho\, v_r\frac{\partial v_r}{\partial r} \neq 0$$

$$\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right)\right] \neq 0$$

WE SEE THAT EVEN FOR A "SIMPLE" SITUATION, ...
TERMS ON LEFT SIDE OF NAVIER-STOKES EQ. ARE NON ZERO

ASIDE) LEFT SIDE OF N.S.

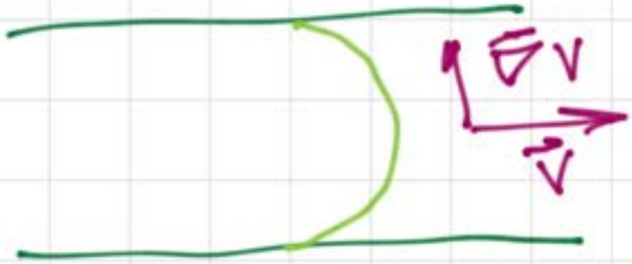
$$\rho \vec{V} \cdot \vec{\nabla} \vec{V}$$

THIS $= 0$ IF $\vec{V} \perp \underbrace{\vec{\nabla}\vec{V}}$

TRUE FOR STRAIGHT LINE FLOW

VELOCITY

GRADIENT OF VELOCITY



$$\vec{V} = (v_x, 0, 0) \qquad \vec{V} \cdot \vec{\nabla} \vec{V} = 0$$

$$\vec{\nabla} V = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\partial v_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

COLUMN VECTOR

$$\vec{V} = (v_x, v_y, 0) \qquad \vec{V} \cdot \vec{\nabla} \vec{V} =$$

$$\vec{\nabla} \vec{V} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & 0 \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \\ v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \\ 0 \end{pmatrix}$$

3X3 ARRAY

CAN WE ESTIMATE (RELATIVE) SIZE OF THESE TERMS?

$$\rho V_r \frac{\partial V_r}{\partial r} \simeq \rho V_r \frac{\Delta V_r}{\Delta r}$$

$$V_r \sim U_0$$

$$\Delta V_r \sim U_0$$

$$\Delta r \sim \underline{2,3\ldots} \; R$$

THUS

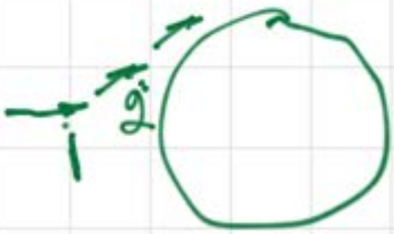
$$\rho V_r \frac{\partial V_r}{\partial r} \simeq -\rho \frac{U_0^2}{R}$$

HOW ABOUT VISCOUS TERM?

$$\mu \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial v_r}{\partial r}$$

ONE PIECE

$$\mu \frac{\partial^2 v_r}{\partial r^2} = \mu \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}\right)$$



$$\simeq \mu \frac{1}{\Delta r}\left(\frac{\Delta v}{\Delta r}\Big|_1 - \frac{\Delta v}{\Delta r}\Big|_2\right)$$

$$\simeq \frac{\mu}{R}\left(\frac{U_0}{R}\right)$$

$$= \mu \frac{U_0}{R^2}$$

TAKE RATIO

$$\frac{\rho U_0^2/R}{\mu U_0/R^2} = \frac{\rho U_0 R}{\mu} = Re$$

WE SURMISE THAT IF
$Re \ll 1$, WE CAN
NEGLECT INERTIA
TERMS RELATIVE TO
VISCOUS TERMS!!

IF SO: EQUATIONS WILL BE
LINEAR.

WE CAN FORMALIZE THIS
BY NONDIMENSIONALIZING
OUR EQUATIONS

EACH TERM IS "COMPARED" TO SCALES
OF PROBLEM.

LENGTH SCALE: R

VELOCITY SCALE: U_0

TIME SCALE: R/U_0

DEFINE:

$$v_r^* = \frac{v_r}{U_0} \qquad v_\theta^* = \frac{v_\theta}{U_0}$$

$$r^* = \frac{r}{R} \qquad r = r^* R$$

SUBSTITUTE DEPENDENT VARIABLES

USE CHAIN RULE ON INDEPENDENT VARIABLES

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial r^*}\frac{\partial r^*}{\partial r} = \Big(\underbrace{\frac{\partial r^*}{\partial r}}_{1/R}\Big)\frac{\partial}{\partial r^*}$$

$$\frac{\partial v_r}{\partial r} = \frac{\partial r^*}{\partial r}\frac{\partial}{\partial r^*}\left(U_0 v_r^*\right)$$

$$= \frac{U_0}{R}\frac{\partial v_r^*}{\partial r^*}$$

$$\frac{\partial^2 v_r}{\partial r^2} = \left(\frac{\partial r^*}{\partial r}\right)^2 U \frac{\partial^2 v_r^*}{\partial r^{*2}}$$

$$= \frac{U}{R^2}\frac{\partial^2 v_r^*}{\partial r^{*2}}$$

WHEN YOU GET DONE

$$v_r^* \frac{\partial v_r^*}{\partial r^*} = O(1)$$

$$\frac{\partial^2 v_r^*}{\partial r^{*2}} = O(1)$$

WE NEED p^*

$$P_c \equiv \frac{\mu U_0}{R} \quad \therefore \; p^* = \frac{p}{\mu U_0 / R}$$

FOR ALL TERMS ...

$$\underbrace{Re\left(\vec{V}^* \cdot \vec{\nabla}^* \vec{V}^*\right)}_{O(1)} = \underbrace{-\vec{\nabla} p^*}_{O(1)} + \underbrace{\nabla^{*2} \vec{V}^*}_{O(1)}$$

THEN IF $Re \to 0$ WE GET

$$\boxed{\vec{\nabla}^* p^* = \nabla^{*2} \vec{V}^*}$$

STOKES EQUATION

LINEAR PDE

MANY SOLUTIONS ARE KNOWN