CBE 30357
12/8/17

FINAL EXAM REVIEW

PRIMARY COURSE TOPICS

1) EXACT SOLUTIONS TO STEADY, "1-D" DIFFERENTIAL FLOW PROBLEMS

2) USE OF MACROSCOPIC MOMENTUM AND BERNOULLI EQUATIONS TO GET FORCES AND DETERMINE FLOWRATE-PRESSURE DROP BEHAVIOR

FOR SIMPLE & COMPLEX DEVICES & SYSTEMS

3) USE OF DIFFERENTIAL EQUATIONS TO ANALYZE FLOWS FOR WHICH $Re \ll 1$

4) ANALYSIS OF TIME-VARYING, DEVELOPING AND BOUNDARY-LAYER FLOWS

1) "EXACT" DIFFERENTIAL ANALYSIS

$\boxed{\vec{V}\cdot\vec{\nabla}V = 0}$

$V_z(R) = 0$

$V_z'(0) = 0$

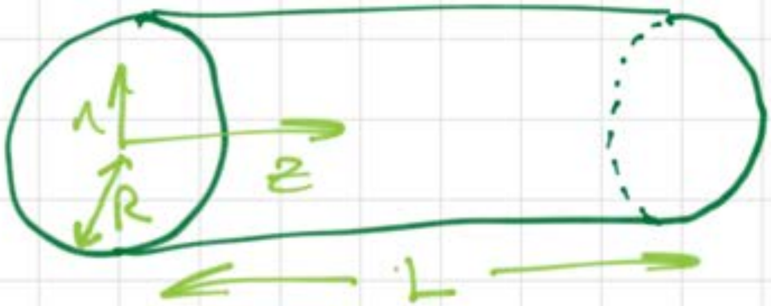


TABLE 3.1

The Conservation of Mass (Continuity Equation)

Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$
Cylindrical coordinates (r, θ, z)	$\cancel{\frac{\partial \rho}{\partial t}} = -\left(\cancel{\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r}} + \cancel{\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta}} + \frac{\partial(\rho v_z)}{\partial z}\right)$
Spherical coordinates (r, θ, ϕ)	$\frac{\partial \rho}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi}\right)$

$$\frac{\partial V_z}{\partial z} = 0$$

CONTRIBUTE THE SAME WAY

$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \rho g_z$$

$P(r) = f(g_r)$

Cylindrical coordinates

r direction

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\cancel{\frac{1}{r}\frac{\partial(r v_r)}{\partial r}}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right] + \rho g_r \qquad (3.3.27a)$$

$P(\theta) = f(g_\theta)$

θ direction

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\cancel{\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \rho g_\theta \qquad (3.3.27b)$$

z direction

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_z}{\partial \theta} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \qquad (3.3.27c)$$

$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\left(\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r}\right)$$

$$\frac{\mu}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} = \frac{\partial P}{\partial z}$$

$$\underset{\text{ONLY}}{f(r)} = \underset{\text{ONLY}}{f(z)} = \underline{\underline{\text{CONST}}}$$

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{r}{\mu}\frac{dP}{dz}$$

$$\int d\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dP}{dz}\int r\,dr$$

$$r\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dP}{dz}\frac{r^2}{2} + C_1$$

$$\int dv_z = \int\left(\frac{1}{\mu}\frac{dP}{dz}\frac{r}{2} + \frac{C_1}{r}\right)dr$$

$$V_z = \frac{1}{\mu}\frac{dP}{dz}\frac{r^2}{4} + C_1 \ln r + C_2$$

$$C_1 = 0 \qquad \left.\begin{array}{l} V_z(r=0) = \text{FINITE} \\ \text{OR } \left.\frac{\partial V_z}{\partial r}\right|_{r=0} = 0 \end{array}\right]$$

GET C_2 FROM $V_z(R)=0$

$$0 = \frac{1}{\mu}\frac{dP}{dz}\frac{R^2}{4} + C_2$$

$$C_2 = -\frac{1}{\mu}\frac{dP}{dz}\frac{R^2}{4}$$

$$V_z(r) = \frac{1}{4\mu}\left(\frac{-\partial P}{\partial z}\right)R^2\left(1 - \frac{r^2}{R^2}\right)$$

WHAT IS WALL SHEAR STRESS?

Cylindrical coordinates

$$\tau_{rr} = 2\mu \cancel{\frac{\partial v_r}{\partial r}} \; 0$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(\cancel{r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)} + \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} \right) \; 0$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\cancel{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r} \right)$$

$$\tau_{\theta\theta} = 2\mu\left(\cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r}{r}} \right)$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left(\cancel{\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}} \right)$$

$$\tau_{zz} = 2\mu \cancel{\frac{\partial v_z}{\partial z}}$$

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

$$\tau_{rz}(R) = -\frac{R}{2}\left(-\frac{\partial P}{\partial z}\right)$$

OR ANY r THAT YOU WANT

WHAT IS RELATION BETWEEN VOLUMETRIC FLOW & PRESSURE DROP?

$$Q = \int_0^{2\pi}\int_0^R v_z(r)\, r\, dr\, d\theta$$

$$Q = \int_0^{2\pi}\int_0^{R} \frac{1}{4\mu}\left(-\frac{\partial P}{\partial z}\right) R^2 \left(1 - \frac{r^2}{R^2}\right) r\,dr\,d\theta$$

$$= \frac{2\pi}{4\mu}\left(-\frac{\partial P}{\partial z}\right) R^2 \left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)_0^R$$

$$\underbrace{\qquad}_{= \frac{r^2}{4}}$$

$$Q = \frac{\pi}{8\mu}\left(-\frac{\partial P}{\partial z}\right) R^4$$

GENERAL CORRELATION

$$f \equiv \frac{\Delta P\, D}{2 \rho L V^2} \qquad V = \text{AVE}(v_z)$$

$$-\frac{\partial P}{\partial z} = \frac{8\mu \pi R^2 V}{\pi R^4}$$

$$\frac{(2R)}{2\rho V^2}\left(-\frac{\partial P}{\partial z}\right) = \frac{2\cancel{R}}{2\rho V^{\cancel{2}}}\left(\frac{8\mu \cancel{V}}{R^{\cancel{2}}}\right) =$$

$2R = D$

$$f = \frac{16}{Re}$$

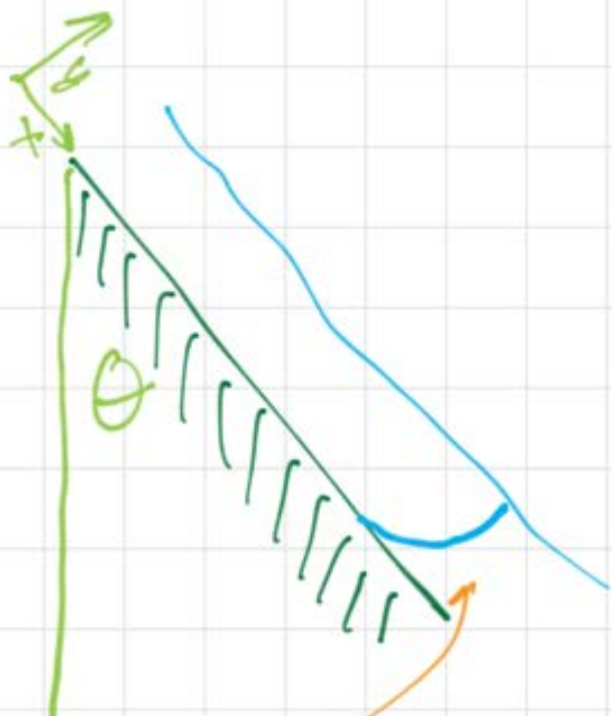
OTHER SITUATIONS

PARABOLIC

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$v_x(-h/2) = 0$$

$$v_x(h/2) = 0$$



$$0 = \mu \frac{\partial^2 v_x}{\partial y^2} + \rho g_x$$

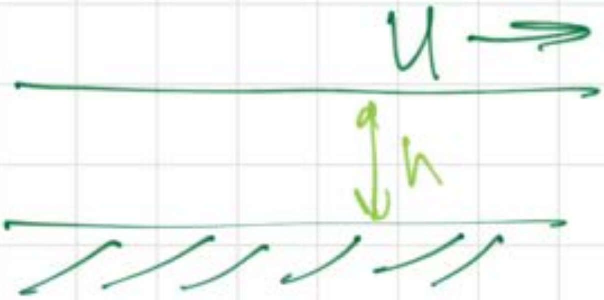
$$g_x = g \cos\theta$$

$$v_x(0) = 0$$

$$\frac{\partial v_x}{\partial y}(y=h) = 0$$

PARABOLIC

QUIESCENT AIR DOES NOT EXERT A SHEAR STRESS

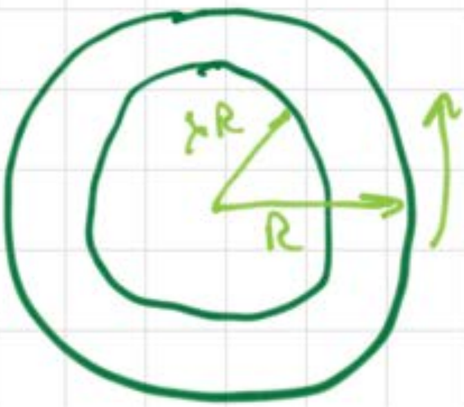


$$0 = \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$V_x(y) = U \frac{y}{h}$$

LINEAR

ROTATING FLOW



$$V_\theta(R) = R\Omega$$

$$V_\theta(\lambda R) = 0$$

$$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)$$

Cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad (3.3.27a)$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad (3.3.27b)$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad (3.3.27c)$$

ONE NON-NEWTONIAN CONSTITUTIVE EQ.

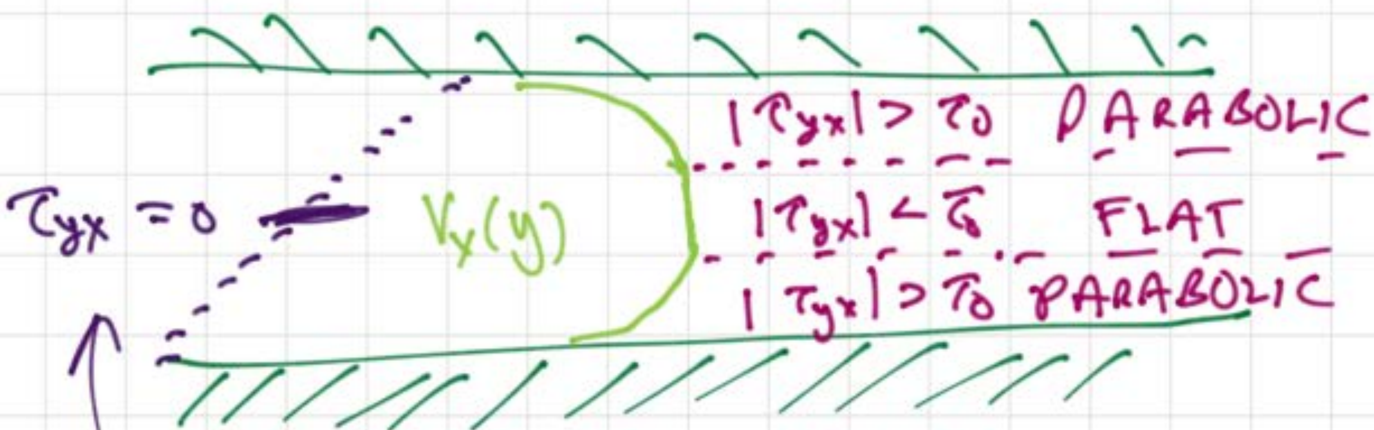
$\dot{\gamma}_x = \frac{\partial V_x}{\partial y}$

BINGHAM PLASTIC

$|\tau_{yx}| < \tau_0 \quad \dot{\gamma}_x = 0$ (NO DEFORMATION)

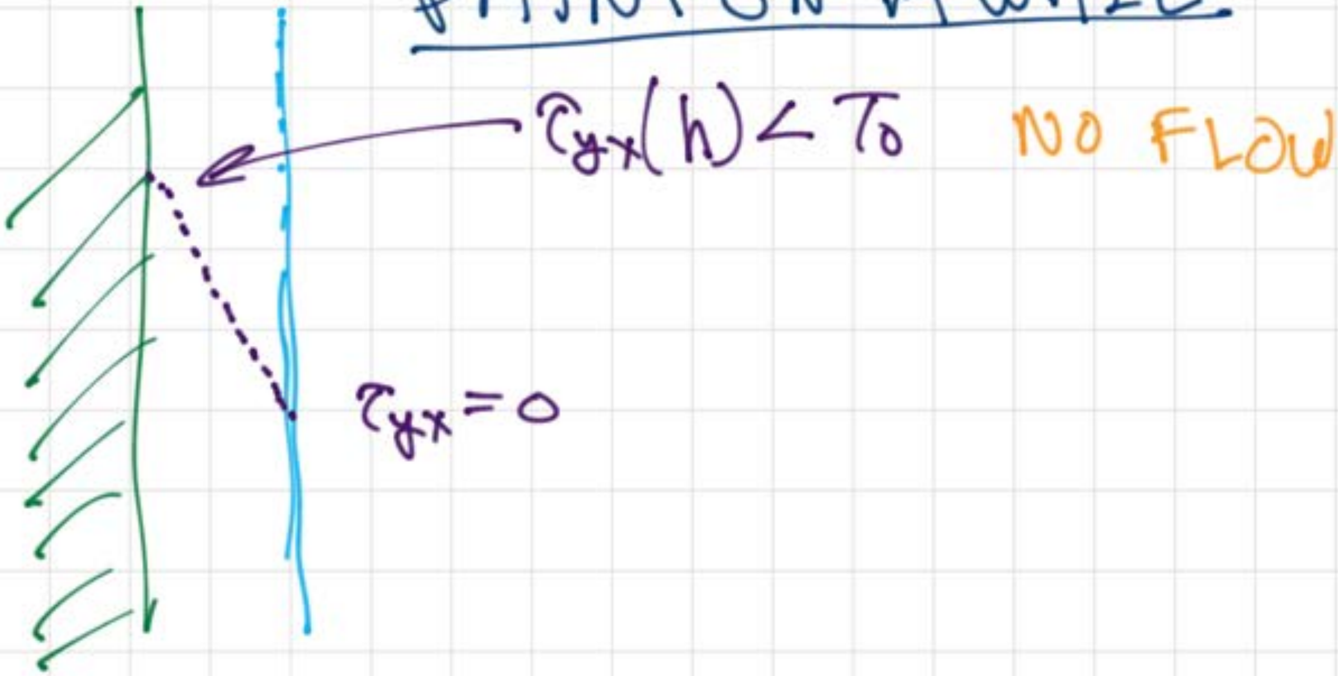
$|\tau_{yx}| > \tau_0$

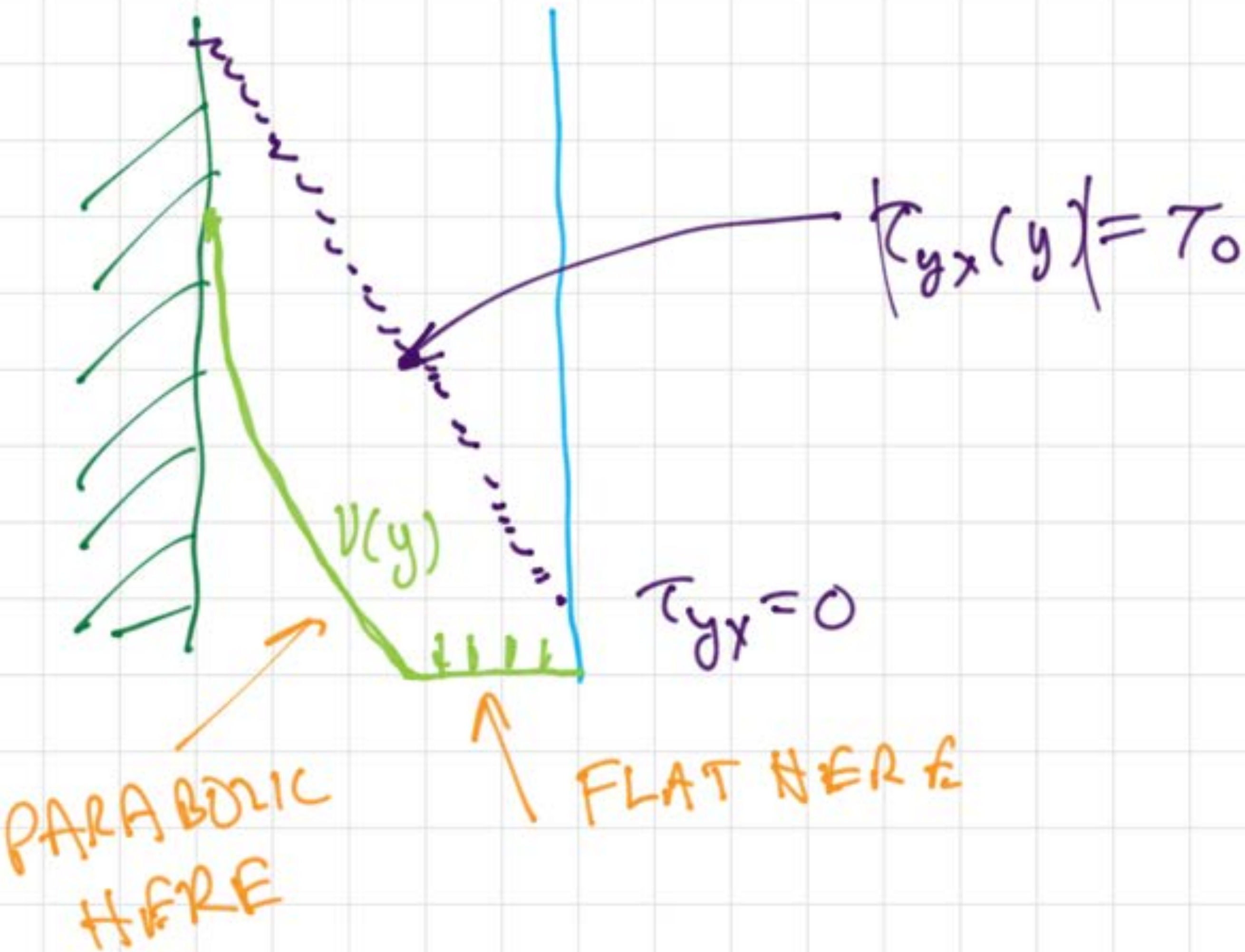
$\tau_{yx} = \pm \tau_0 + \mu_0 \dot{\gamma}_x$



SHAPE OF STRESS PROFILE IS INDEPENDENT OF FLUID!!

PAINT ON A WALL





2) MACROSCOPIC BALANCES FOR MORE COMPLEX PROBLEMS

MASS $\frac{dM}{dt} = \sum_i \rho A V_i$

$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$

$V_2 = \frac{V_1 A_1}{A_2}$ SPEEDS UP

MOMENTUM
- TO GET FORCES!!
- VECTOR DIRECTION IS CRITICAL

EQ COMES FROM VOLUME INTEGRATION OF DIFFERENTIAL EQUATIONS

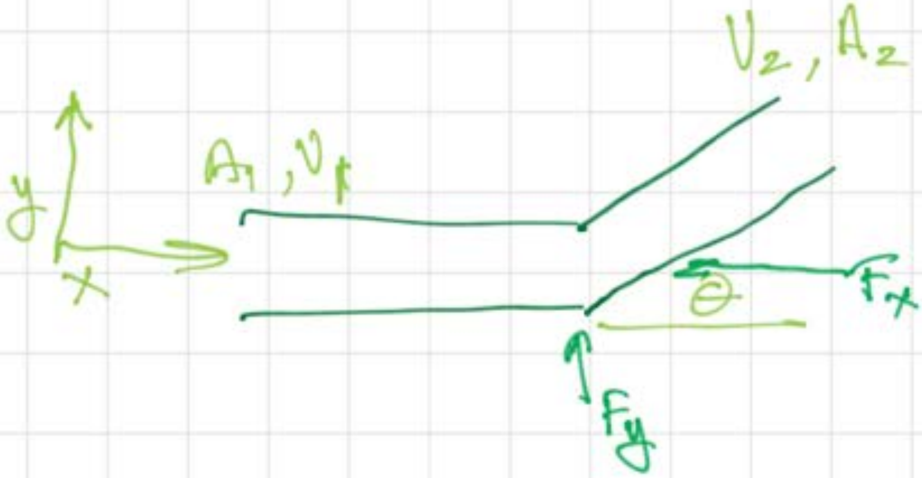
SIGN FROM

FLOW TERM $\int (\rho \bar{V})(\bar{V}\cdot\bar{n})\,ds \Rightarrow \rho V_1 V_1 A_1$

x $\quad -\rho V_1 V_1 A_1 + \rho V_2 V_2 A_2 \cos\theta = P_1 A_1 - P_2 A_2 \cos\theta + F_x$

y $\quad +\rho V_2 V_2 \sin\theta = -P_2 A_2 \sin\theta + F_y$

KEEP DIRECTIONS SEPARATE !!



$$V_1 A_1 = V_2 A_2$$

$$-\rho V_1 V_1 A_1 + \rho V_2 V_2 A_2 \cos\theta = P_1 A_1 - P_2 A_2 \cos\theta + F_x$$

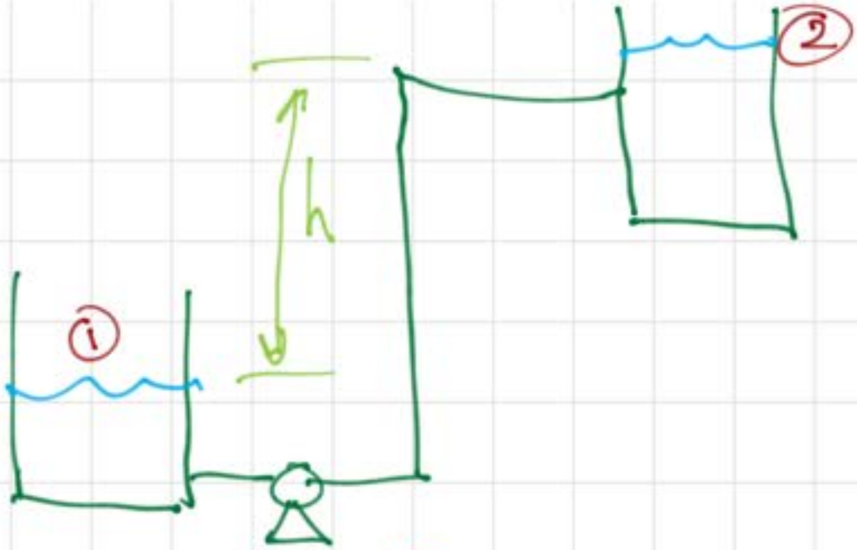
$$\rho V_2 V_2 A_2 \sin\theta = -P_2 A_2 \sin\theta + F_y$$

$$\left(\frac{V_2^2}{2} + \frac{P_2}{\rho}\right) - \left(\frac{V_1^2}{2} + \frac{P_1}{\rho}\right) = -h_v = -\tfrac{1}{2} k V_2^2$$

BY KNOWING $A_1, A_2, \rho, V_1, P_1, K$

CAN GET:

$P_2, V_2, F_x, F_y,$



$$\left(\cancel{\frac{V_2^2}{2}}^{\,0} + \frac{P_2}{\rho} + g h_2\right) - \left(\cancel{\frac{V_1^2}{2}}^{\,0} + \frac{P_1}{2} + g h_1\right) = \delta W_s - l_v$$

$P_2, P_1 = P_{ATM}$

$$g\left(h_2 - h_1\right) = \delta W_s - \underbrace{\sum_i \frac{k}{2} V_i^2}_{\text{FITTINGS}} +$$

$$- \underbrace{\sum_i \frac{2 L_i f_i V_i^2}{D_i}}_{\text{PIPE SECTIONS}}$$

f_i: FRICTION FACTOR

$$\dot{m}\,\delta W_s = \dot{W}_s \quad \text{(POWER INPUT)}$$

$\dot{m}$: MASS FLOW RATE

$\dot{W}_s$: WHAT WE WANT TO KNOW

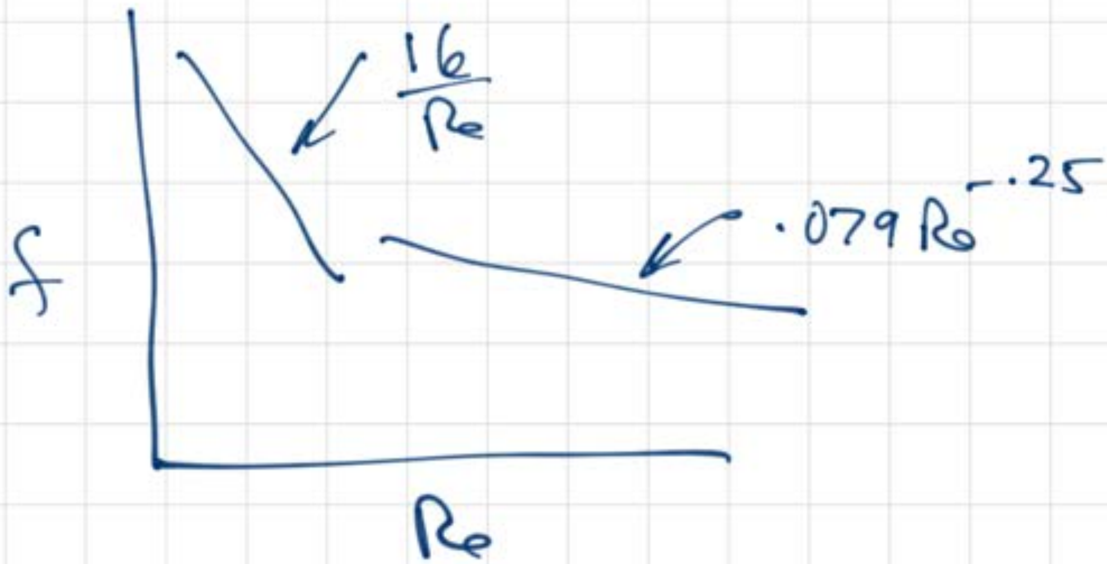
HYDRAULIC DIAMETER

$$D_H = \frac{4\ \text{AREA}}{\text{PERIMETER}}$$

$$Re = \frac{\rho V D_H}{\mu}$$

$$V = \frac{Q}{\text{AREA}}$$

REAL AVERAGE VELOCITY



3) $R_e << 1$

- VISCOUS FORCES DOMINATE
- CAN NEGLECT INERTIA TERMS BECAUSE THEY ARE MUCH SMALLER THAN VISCOUS TERMS

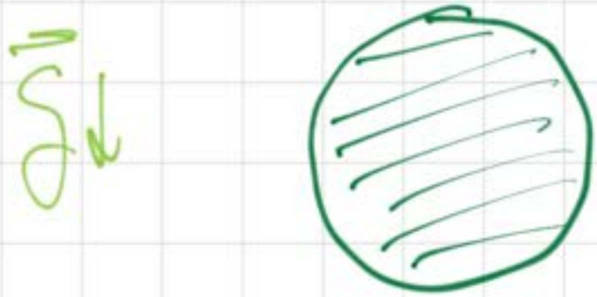
FROM SCALING

$$\left.\begin{aligned} \bar{\nabla} \cdot \bar{v} &= 0 \\ 0 &= -\bar{\nabla} P + \mu \nabla^2 \bar{v} \end{aligned}\right]$$

(ALWAYS AT A PSEUDO-STEADY-STATE)

- MANY SOLUTIONS EXIST.

STILL NOT TRIVIAL



$$\sum F = \text{GRAVITY} + \text{BUOYANCY} + \text{DRAG}$$

$$\sum F = \rho_s g V - \rho_f g V - 6\pi\mu U R$$

$$= \rho_s g \tfrac{4}{3}\pi R^3 - \rho_f g \tfrac{4}{3}\pi R^3 - 6\pi\mu U R$$

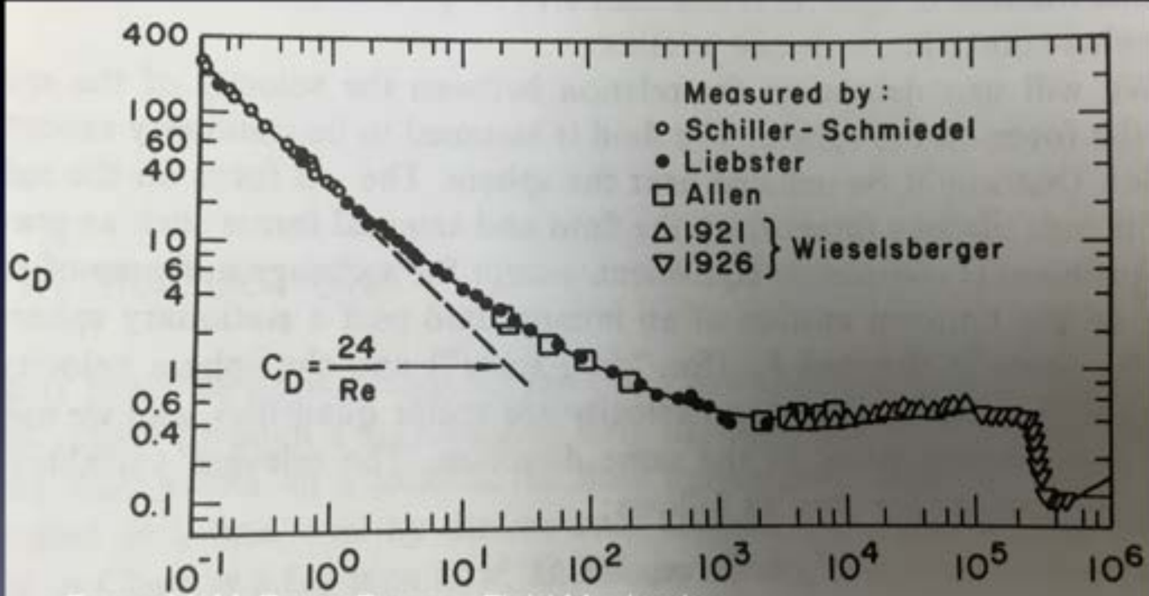
$$6\pi\mu U R = \tfrac{4}{3}(\rho_s - \rho_f) g \pi R^3$$

$$U = \frac{2}{9}\frac{(\rho_p - \rho_f)}{\mu} g R^3$$

Drag coefficient

Reynolds number: $Re = \frac{D_p V_p \rho}{\eta}$

drag coefficient: $C_D = \frac{8}{\pi} \frac{F_D}{\rho V_p^2 D_p^2}$



From: M. M. Denn, Process Fluid Mechanics

Figure 4-1. Drag coefficient as a function of Reynolds number for flow past a sphere. (Reproduced from H. Schlichting, *Boundary Layer Theory*, 6th ed., McGraw-Hill Book Company, New York, 1968, by

CAN EXTEND TO LARGE MOLECULES (OR SMALL PARTICLES)

BOLTZMAN

$$D = \frac{k_B T}{6 \pi \mu R}$$

PARTICLE DIFFUSIVITY

PARTICLE RADIUS

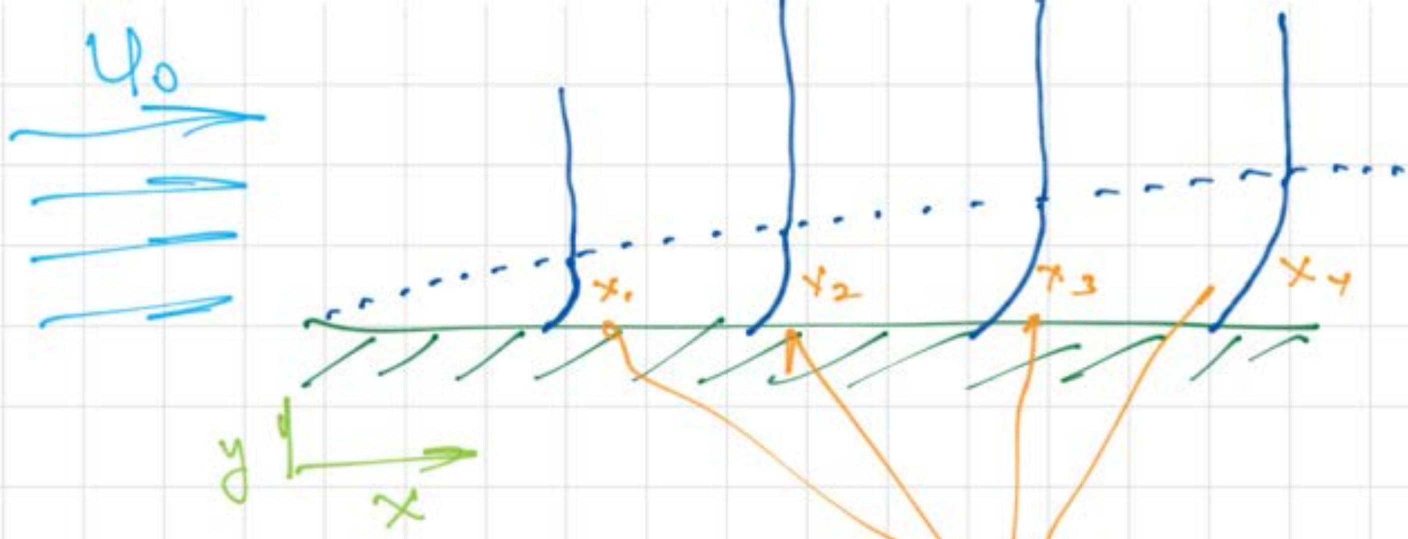
4) TIME VARYING, DEVELOPING, BOUNDARY-LAYER FLOWS

FOR AN INFINITY BOX, NO GEOMETRIC LENGTH SCALE

$$\therefore \quad \delta \sim \ell \sim \sqrt{\nu t}$$

FOUNDATION OF SELFSIMILARITY

$$\eta \equiv y/\sqrt{4\nu t}$$



ALSO SELF SIMILAR

$$\eta = \frac{y}{\sqrt{\nu x / U_0}}$$

BOUNDARY-LAYER EQUATIONS

$$\boxed{\begin{aligned} &\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \\ &v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2} \\ &\frac{\partial p}{\partial y} = 0 \end{aligned}}$$

PRESSURE IS FROM: $\Delta\left(\frac{V^2}{2} + gh + \frac{p}{\rho}\right) = 0$

PDE $\Rightarrow$ ODE

NUMERICAL SOLUTION GIVES

δ, τ_w

LUBRICATION

EITHER SLIGHTLY NON-PARALLEL FLOW OR SQUEEZE FILM, CAUSE FLOW OF LIQUID THROUGH A SMALL PASSAGE THAT BUILDS UP PRESSURE THAT SUPPORTS LOAD....

SEE NOTES + H.W.