CBE 30357

9/5/17

CONTINUED DERIVATION OF DIFFERENTIAL EQUATIONS FOR FLUID FLOW

RECALL:

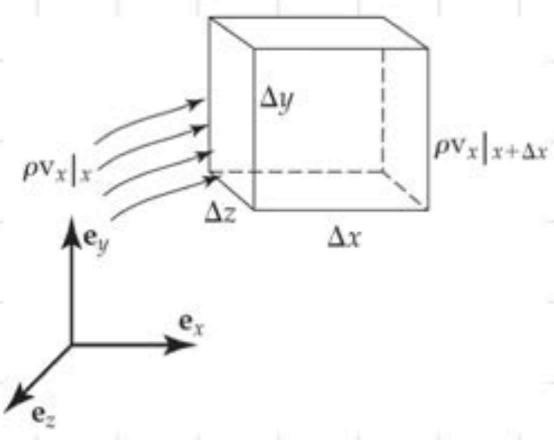
WE OBTAINED A DIFFERENTIAL EQUATION FOR MASS BY APPLYING PRINCIPLE OF MASS CONSERVATION TO (ARBITRARY) DIFFERENTIAL

CONTROL VOLUME

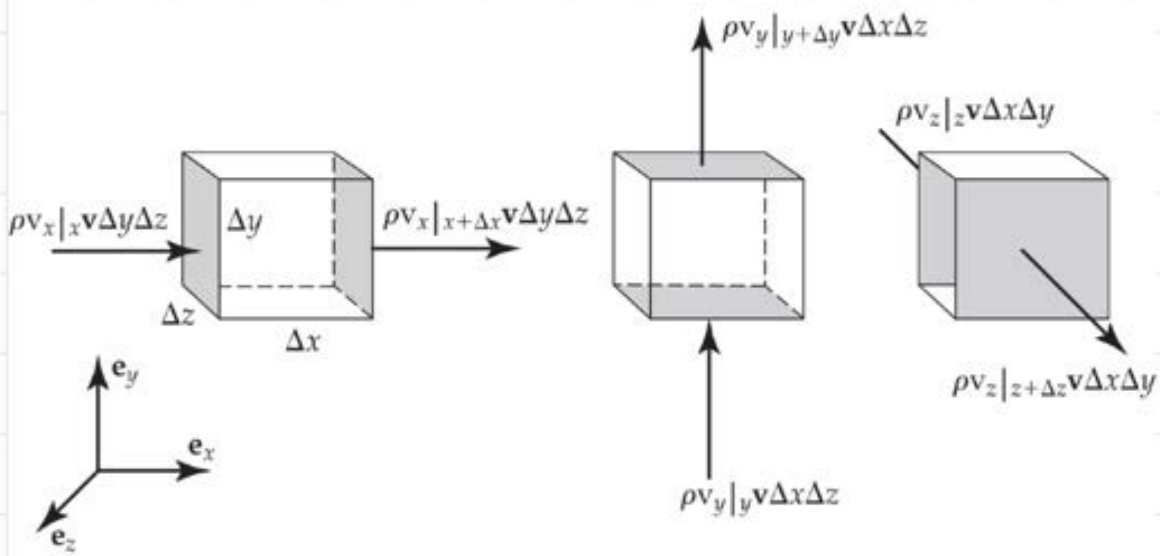
$$\frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho = -\rho\vec{\nabla}\cdot\vec{v}$$

IF ρ = CONST

$$\vec{\nabla}\cdot\vec{v} = 0$$



WE FOLLOWED THE SAME PROCEDURE FOR MOMENTUM CONSERVATION: $\frac{d\vec{p}}{dt} = \Sigma \vec{F}$



$\frac{\text{MOMENTUM}}{\text{VOLUME}} \times$ VOLUMETRIC FLOW $\Rightarrow$ FOR EACH FACE

(x) • $\rho\bar{v}$ | $v_x \Delta y \Delta z$

FULL VECTOR MOMENTUM | ↑ SCALAR COMPONENT

CONSERVED QUANTITY | CARRIES MOMENTUM IN

(z) • $\rho\vec{v}$ | $v_z \Delta x \Delta y$

Using the differential cube formalism and constructing the inflow and outflow terms as CQ/ volume * volumetric flow we then shrink the cube to 0 and obtain the following (completely general) intermediate result:

$$dV\left[\underbrace{\frac{\partial \rho \bar{v}}{\partial t} + \bar{\nabla}\cdot(\rho \bar{v}\bar{v})}_{\text{EXPRESSION FOR TRANSPORT BY FLUID FLOW AND CHANGE WITH TIME}}\right] = \Sigma \bar{F}$$

ALL ACCELERATION EFFECTS

USING CONTINUITY:

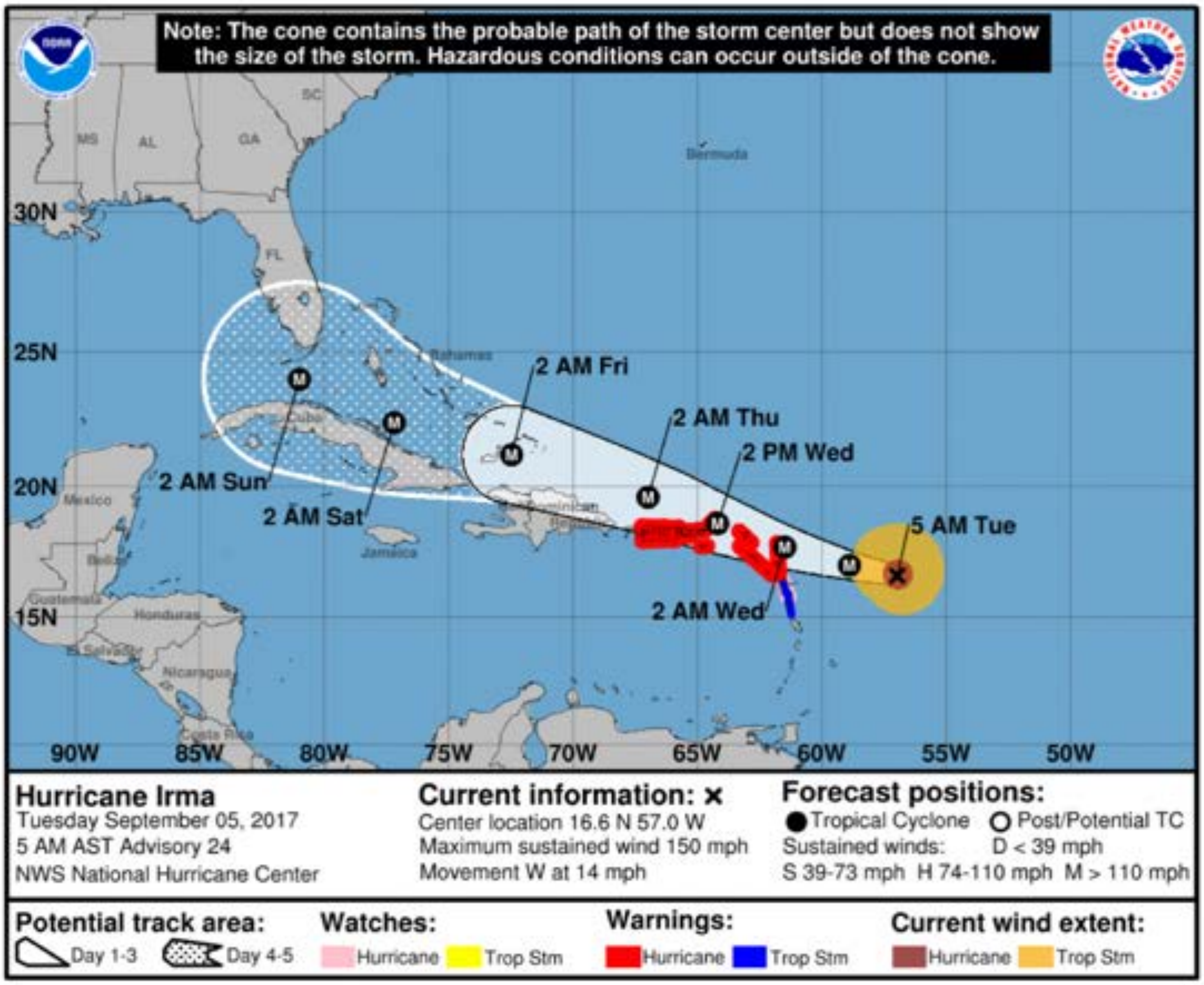
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0$$

WE GET:

$$dV \left[\underbrace{\rho \frac{\partial \vec{V}}{\partial t} + \rho \vec{V} \cdot \vec{\nabla} \vec{V}}_{\text{FINAL FORM}} \right] = \Sigma \vec{F}$$

NOW WE NEED TO CONSIDER FORCES

MOTIVATION !!



Note: The cone contains the probable path of the storm center but does not show the size of the storm. Hazardous conditions can occur outside of the cone.
2 AM Fri
2 AM Thu
2 PM Wed
5 AM Tue
2 AM Wed
2 AM Sun
2 AM Sat
30N
25N
20N
15N
90W
85W
80W
75W
70W
65W
60W
55W
50W
Hurricane Irma
Tuesday September 05, 2017
5 AM AST Advisory 24
NWS National Hurricane Center
Current information: x
Center location 16.6 N 57.0 W
Maximum sustained wind 150 mph
Movement W at 14 mph
Forecast positions:
Tropical Cyclone
Post/Potential TC
Sustained winds: D < 39 mph
S 39-73 mph H 74-110 mph M > 110 mph
Potential track area:
Day 1-3
Day 4-5
Watches:
Hurricane
Trop Stm
Warnings:
Hurricane
Trop Stm
Current wind extent:
Hurricane
Trop Stm

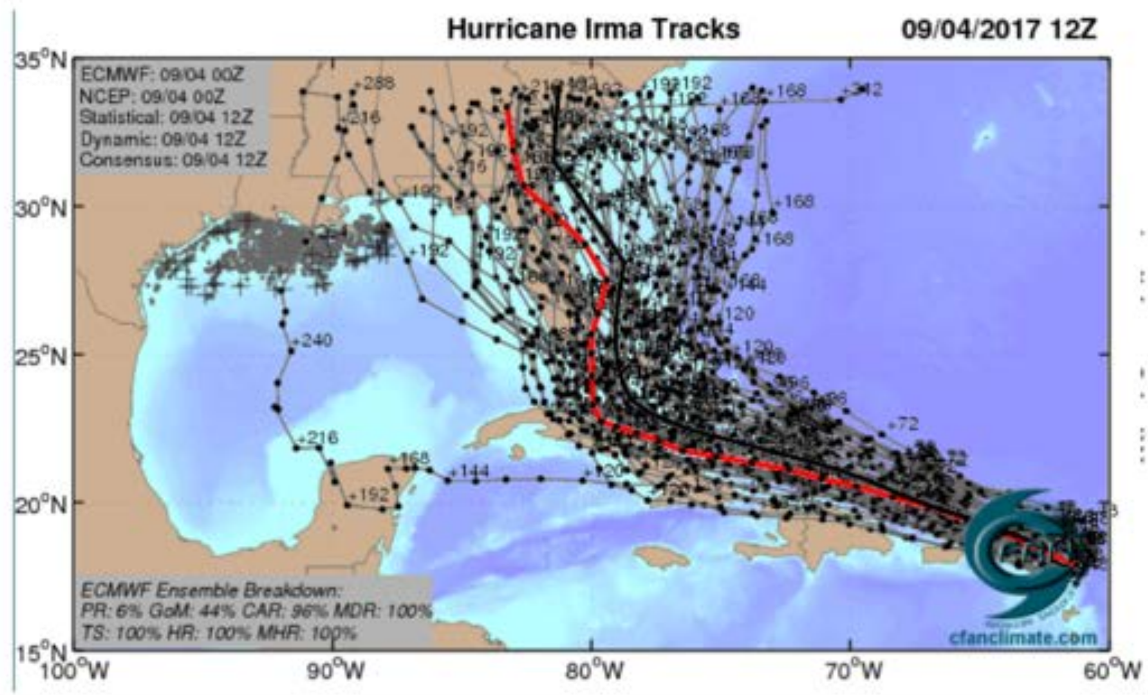
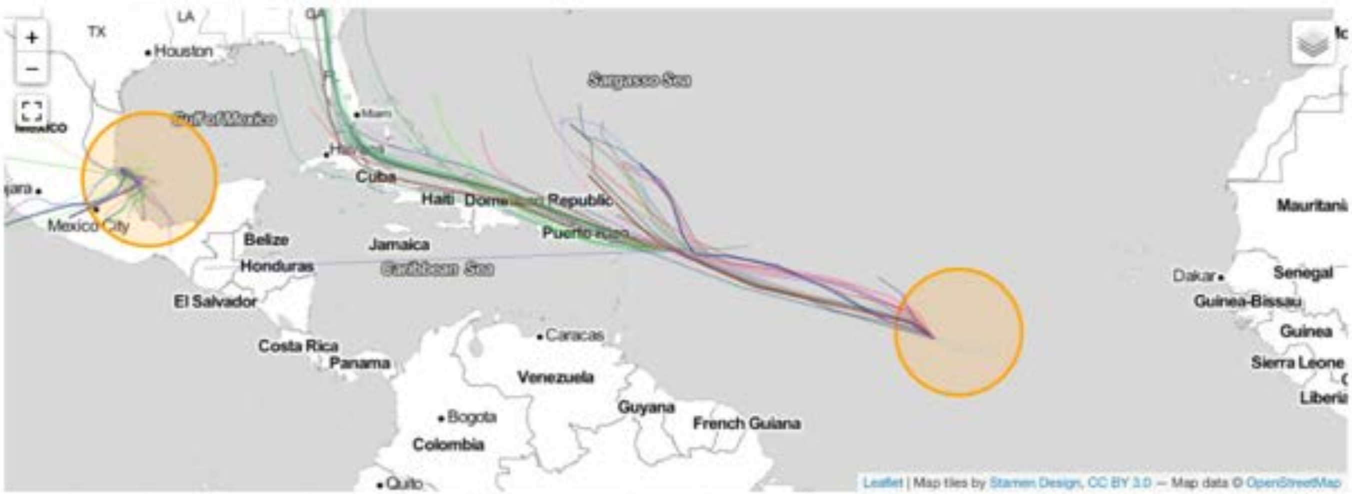


Figure 5. The 0Z September 4, 2017, track forecast by the operational European model for Irma (red line, adjusted by CFAN using a proprietary technique that accounts for storm movement since 0Z), along with the track of the average of the 50 members of the European model ensemble (heavy black line), and the 50 track forecasts from the 0Z Monday European model ensemble forecast (grey lines). Image credit: CFAN.

Irma Spaghetti Models



The uncertainty in the exact wind speed, barometric pressure, humidity at every location anywhere close to the storm, combined with the limit on “resolution”, the spatial grid scale for the numerical solution and that the fluid flow equations are nonlinear — meaning that a small change or error could have either a small or incommensurately large effect, leads to the uncertainty in the path shown here.

Still this level of forecast is much better than guess basing in past hurricane paths!

TABLE 3.4

(Continued)

Spherical coordinates

r direction

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = -\frac{\partial p}{\partial r} + \rho g_r$$
$$+ \mu\left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2} - 2\frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta - \frac{2}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right] \tag{3.3.28a}$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta v_r}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta \tag{3.3.28b}$$

CORIOLIS

ϕ direction

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\phi \tag{3.3.28c}$$

Task today

Continue the derivation to include forces

1. Body forces (e.g., Gravity)
2. Surface forces (pressure, shear stresses that arise because of the fluid viscosity)

Complication of surface forces is that we will not only have two kinds of forces (pressure, viscous), but that we need to always consider separately the normal and tangential components of the forces ---

This leads to 3 by 3 array of force components
" Stress Tensor"

Forces:

1) Body Forces. E.g., Gravity $\vec{E}, \vec{B}$

2) Surface forces... Pressure and shear stress.

BODY FORCE:

$\frac{\text{FORCE}}{\text{VOLUME}} \times \text{VOLUME}$

$\rho \vec{g} \quad \Delta x \Delta y \Delta z$

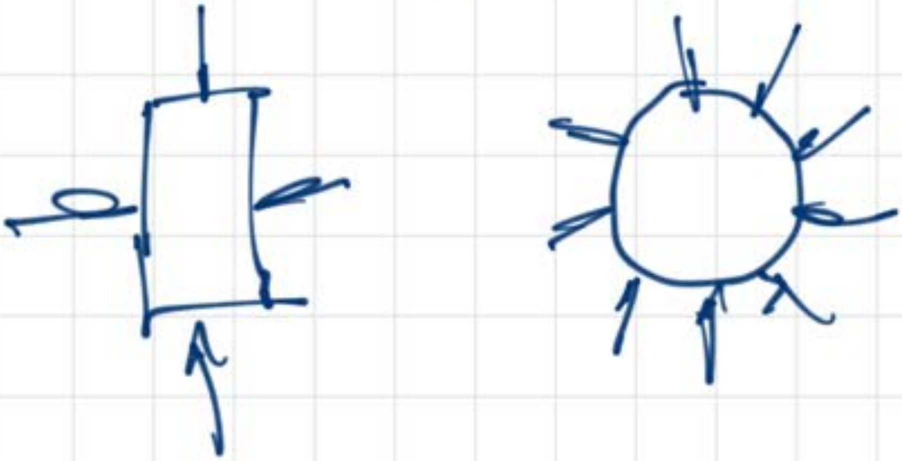
(SAVE FOR LATER)

SURFACE FORCES

PRESSURE IS ISOTROPIC:
ACTS NORMAL TO SURFACE

TO DISCERN EFFECT: NEED ORIENTATION OF SURFACE

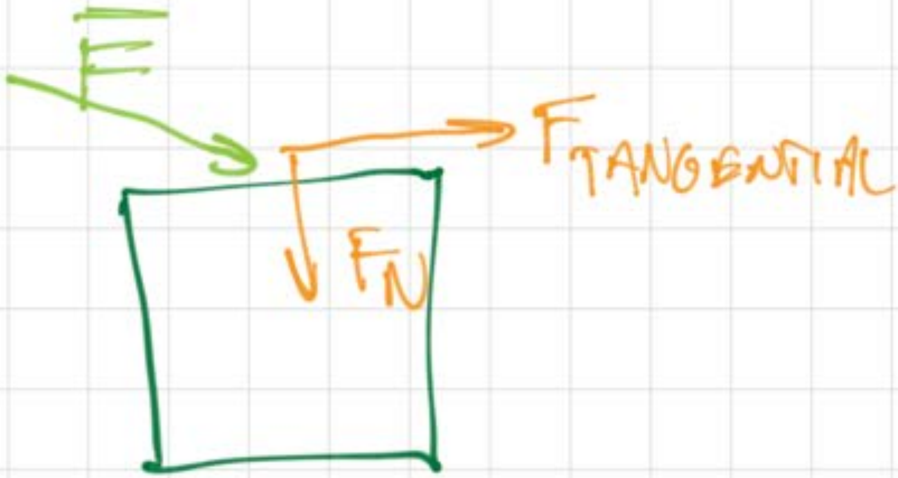
p



FOR AN ARBITRARY FORCE

PRESSURE + SHEAR

NORMAL + TANGENTIAL



$\vec{F}$ CAUSES TRANSLATION + ROTATION

BUT FOR EXACTLY THE SAME FORCE

WE ALWAYS NEED SURFACE ORIENTATION

$F_T = 0$

TRANSLATION IN DIFFERENT DIRECTION (NO APPARENT ROTATION)

THIS HAS ALL BEEN FIGURED OUT

$$\bar{\sigma} = \bar{\tau} - \bar{\bar{I}}p$$

STRESS TENSOR

VISCOUS STRESSES

PRESSURE

"TENSOR" (2ND ORDER TENSOR)

A vector is a 1st order tensor
A scalar is a zero order tensor

$$\bar{\bar{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}$$

$$\bar{\bar{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

IDENTITY TENSOR

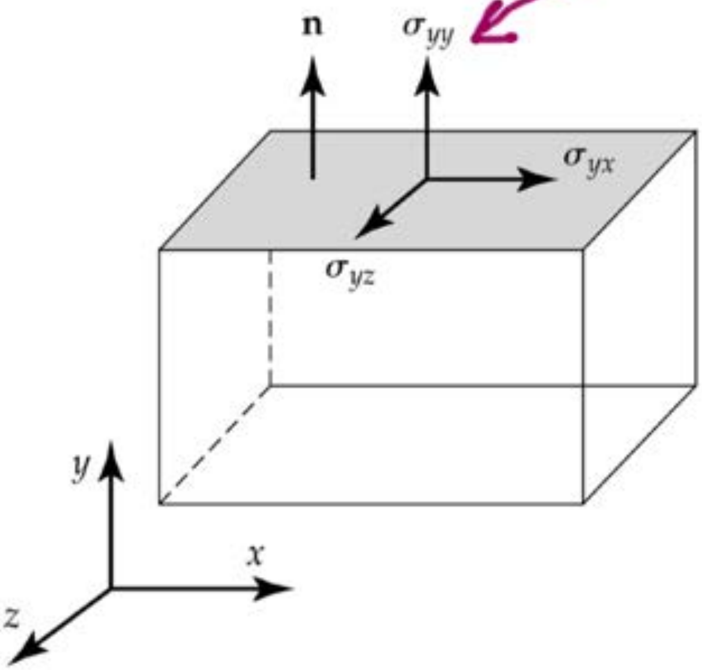
$$\bar{\bar{\sigma}} = \sigma_{ij} = \begin{vmatrix} \tau_{xx} - p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} - p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} - p \end{vmatrix}$$

"INDEX" NOTATION

SURFACE FORCES

NORMAL TO SURFACE

Figure 2.9 Cubic fluid element showing the stresses acting on a face of constant y. As shown, the normal stress σ_{yy} is tensile.



σ_{yy}
NORMAL FORCE

PEARSON

Transport Phenomena in Biological Systems, Second Edition
George A. Truskey, Fan Yuan, and David F. Katz

σ_{yy}

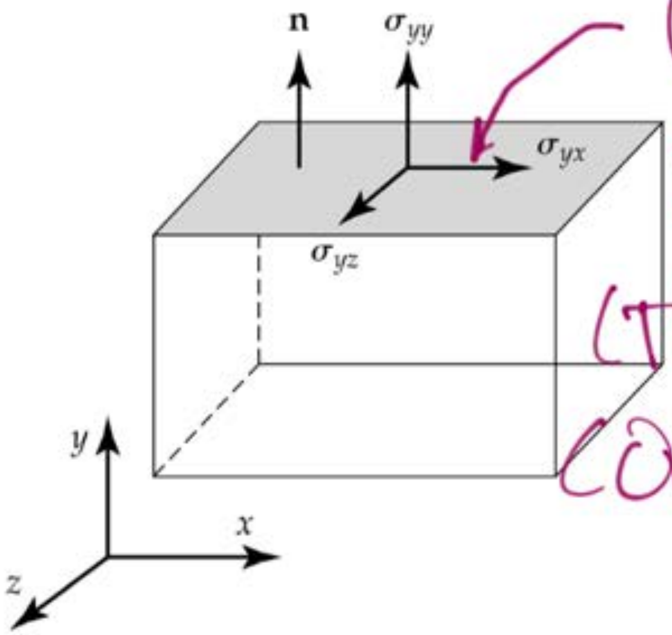
FACE
(ORIENTATION)

DIRECTION
(OF FORCE)

SURFACE FORCES

TANGENTIAL TO SURFACE

Figure 2.9 Cubic fluid element showing the stresses acting on a face of constant y. As shown, the normal stress σ_{yy} is tensile.



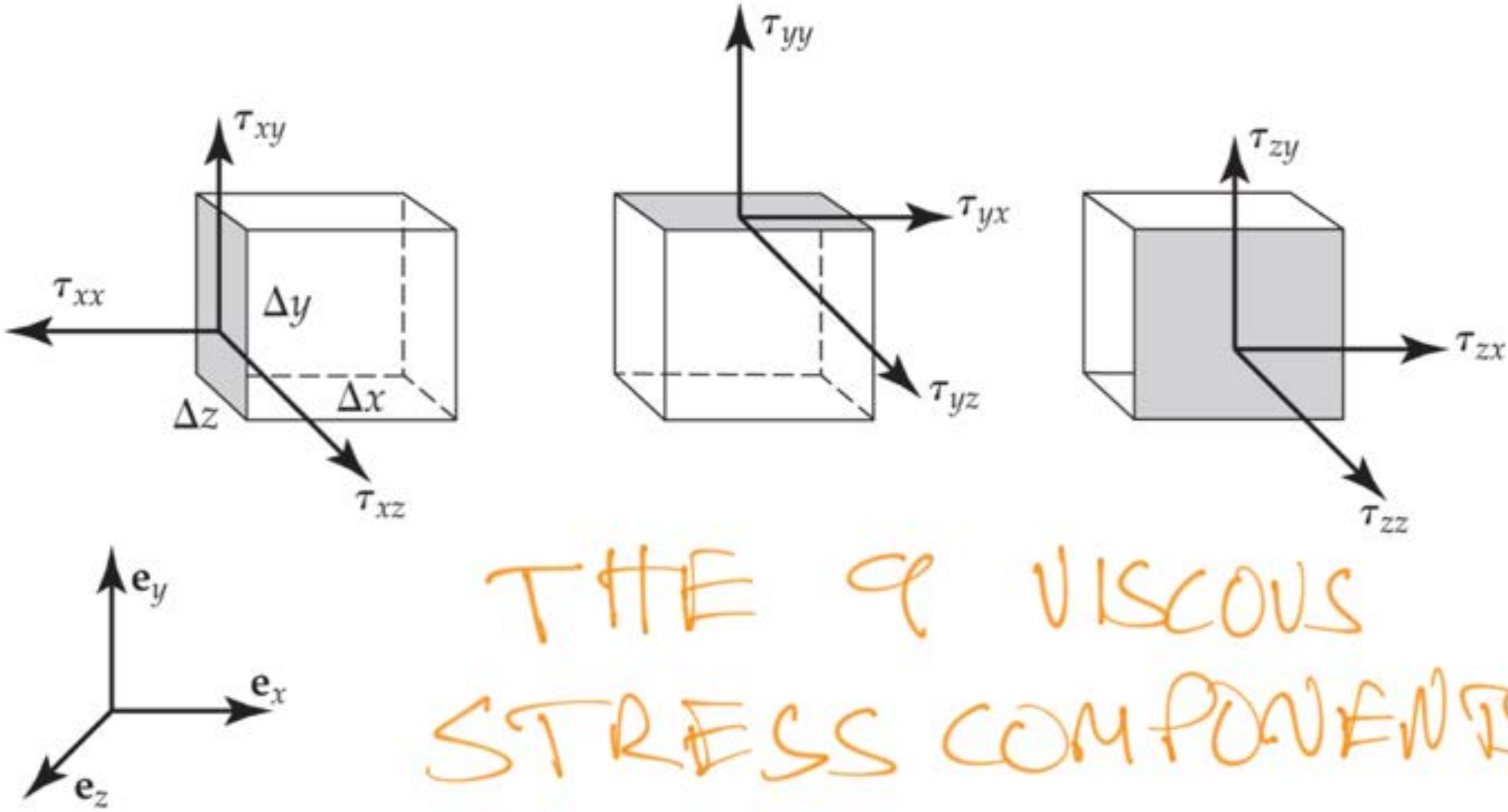
σ_{yx} "SHEAR" (TANGENTIAL) COMPONENT

PEARSON *Transport Phenomena in Biological Systems*, Second Edition
George A. Truskey, Fan Yuan, and David F. Katz

σ_{yx} ← DIRECTION

FACE

Figure 3.5 Schematic of components of the viscous stress tensor.



THE 9 VISCOUS STRESS COMPONENTS

$$\bar{\bar{\sigma}} = \bar{\bar{\tau}} - \bar{\bar{I}}p$$

$$\bar{\bar{\sigma}} = \sigma_{ij} = \begin{vmatrix} \tau_{xx} - p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} - p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} - p \end{vmatrix}$$

"INDEX" NOTATION

COMPLETE EXPRESSION FOR SURFACE FORCES

$$\overline{\overline{\sigma}} = \overline{\overline{\tau}} - \overline{\overline{I}}\,p$$ PRESSURE

ALL SURFACE FORCES

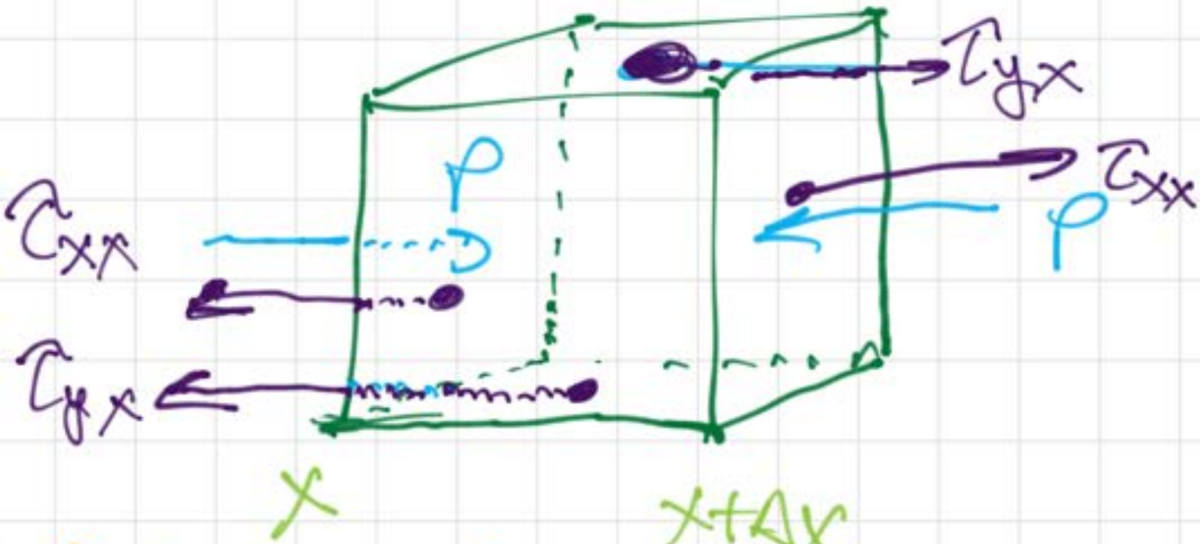
VISCOUS SURFACE FORCES

$$\overline{\overline{\sigma}} = \sigma_{ij} = \begin{pmatrix} \tau_{xx} - p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} - p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} - p \end{pmatrix}$$

TANGENTIAL COMPONENTS

NORMAL COMPONENTS

FORCE BALANCE FOR CUBE



X-DIRECTION:

$$(\tau_{xx} - p)\big|_{x+\Delta x}\Delta y \Delta z - (\tau_{xx} - p)\big|_{x}\Delta y \Delta z$$

$$\tau_{yx}\big|_{y+\Delta y}\Delta z \Delta x - \tau_{yx}\big|_{y}\Delta z \Delta x$$

$$\tau_{zx}\big|_{z+\Delta z}\Delta x \Delta y - \tau_{zx}\big|_{z}\Delta x \Delta y$$

SURFACE IN X-DIRECTION

$$F_{Sx} = -\left(P_{x+\Delta x} - P_x\right)\Delta y \Delta z +$$

$$\left(\tau_{xx}\big|_{x+\Delta x} - \tau_{xx}\big|_x\right)\Delta y \Delta z +$$

$$\left(\tau_{yx}\big|_{y+\Delta y} - \tau_{yx}\big|_y\right)\Delta z \Delta x +$$

$$\left(\tau_{zx}\big|_{z+\Delta z} - \tau_{zx}\big|_z\right)\Delta x \Delta y$$

$\tilde{\ }\ \tilde{\ }\ \tilde{\ }\ \tilde{\ }$

F_{Sy}, F_{Sz}

PUTTING THIS ALL TOGETHER...

$$dV\left(\rho\frac{\partial \vec{v}}{\partial t} + \rho\,\vec{v}\cdot\vec{\nabla}\vec{v}\right) =$$

$$dV\left(-\vec{\nabla}P + \vec{\nabla}\cdot\bar{\bar{\tau}} + \rho\vec{g}\right)$$

WHICH GIVES...

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho\,\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}P + \vec{\nabla}\cdot\bar{\bar{\tau}} + \rho\vec{g}$$

CAUCHY - MOMENTUM EQS

Conservation of Linear Momentum

Rectangular coordinates

x component

$$\rho\left[\frac{\partial \mathrm{v}_x}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_x}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_x}{\partial z}\right] = \rho g_x - \frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

y component

$$\rho\left[\frac{\partial \mathrm{v}_y}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_y}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_y}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_y}{\partial z}\right] = \rho g_y - \frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]$$

z component

$$\rho\left[\frac{\partial \mathrm{v}_z}{\partial t} + \mathrm{v}_x\frac{\partial \mathrm{v}_z}{\partial x} + \mathrm{v}_y\frac{\partial \mathrm{v}_z}{\partial y} + \mathrm{v}_z\frac{\partial \mathrm{v}_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

"CAUCHY-MOMENTUM EQUATIONS"

IF WE LOOK CAREFULLY

1 – MASS BALANCE

3 – MOMENTUM COMPONENTS

HOWEVER EVEN THOUGH

$$\tau_{xy} = \tau_{yx}$$

THERE ARE STILL TOO MANY UNKNOWNS

WE NEED CONSTITUTIVE EQUATIONS

"EXPERIMENTALLY VERIFIED MATHEMATICAL RELATIONS BETWEEN VELOCITY COMPONENTS AND STRESS COMPONENTS"

WE START WITH:

"NEWTONIAN FLUID"

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

(PLUS A BIT MORE) ...

THE RESULT IS THE NAVIER STOKES EQ'S.

Shear-Stress Tensor for an Incompressible Newtonian Fluid

Rectangular coordinates

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} \tag{3.3.22a}$$

$$\tau_{yx} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \tag{3.3.22b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \tag{3.3.22c}$$

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} \tag{3.3.22d}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \tag{3.3.22e}$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \tag{3.3.22f}$$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

TABLE 3.1

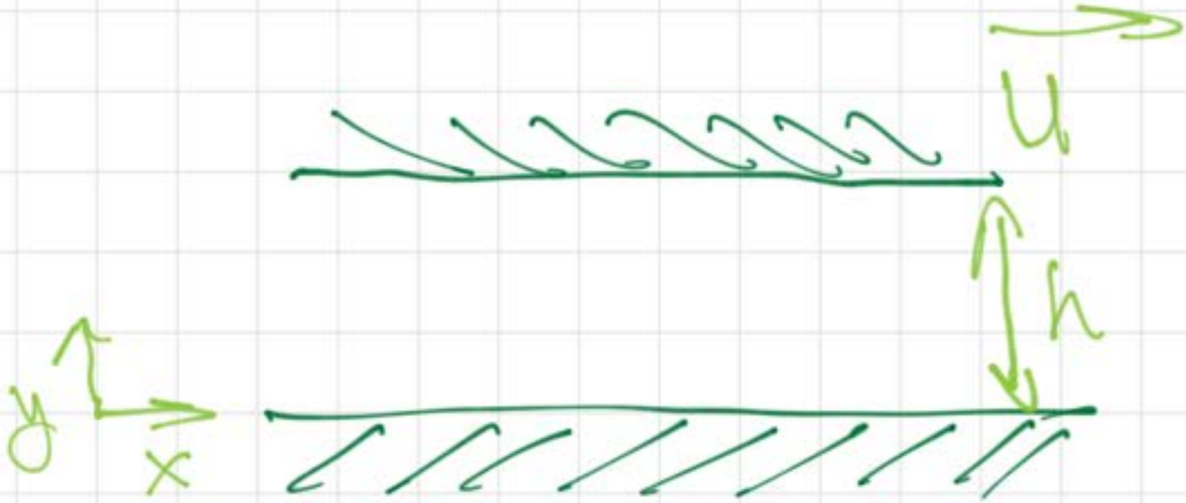
The Conservation of Mass (Continuity Equation)

Rectangular coordinates (x, y, z) $$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$$

These equations can be used to solve almost any fluid flow problem if the fluid is Newtonian.

Our challenge is to learn to use them!

Let's start with simplest problem



Bottom plate is fixed. Top moves with velocity $\vec{U}$.

WHAT IS $V_x(y)$?

WHAT FORCE IS REQUIRED?

NEWTONIAN FLUID, μ

$\rho = \text{CONST}$

LOOK AT CONTINUITY

TABLE 3.1

The Conservation of Mass (Continuity Equation)	
Rectangular coordinates (x, y, z)	$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

BUT $\frac{\partial}{\partial z} = 0$

NOTHING CHANGES IN X, $\frac{\partial}{\partial x} = 0$

$$\therefore \frac{\partial v_y}{\partial y} = 0$$

ONLY $V_x(y)$

TABLE 3.4

Navier–Stokes Equation for an Incompressible Fluid

Rectangular coordinates

x direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \boxed{\frac{\partial^2 v_x}{\partial y^2}} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

y direction

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

AFTER SOME WORK

$$0 = \mu\frac{\partial^2 v_x}{\partial y^2}$$

$$v_x(0) = 0, \quad v_x(h) = U$$

SOLVE

$$0 = \mu \frac{d}{dy}\left(\frac{dv_x}{dy}\right)$$

$$\int d\left(\frac{dv_x}{dy}\right) = \int 0$$

$$\frac{dv_x}{dy} - c_1 = 0$$

$$\int d(v_x) = \int c_1\, dy$$

$$v_x = c_1 y + c_2$$

FIT B.C.'S

$$V_x(0) = 0 \quad \therefore C_2 = 0$$

$$V_x(h) = U = C_1 h$$

$$\therefore C_1 = \frac{U}{h}$$

$$\therefore \boxed{V_x(y) = \frac{y}{h} U}$$

LINEAR PROFILE

μ DIVIDES OUT

AND SO DOES NOT

AFFECT PROFILE.

(μ DOES AFFECT THE FORCE!!)