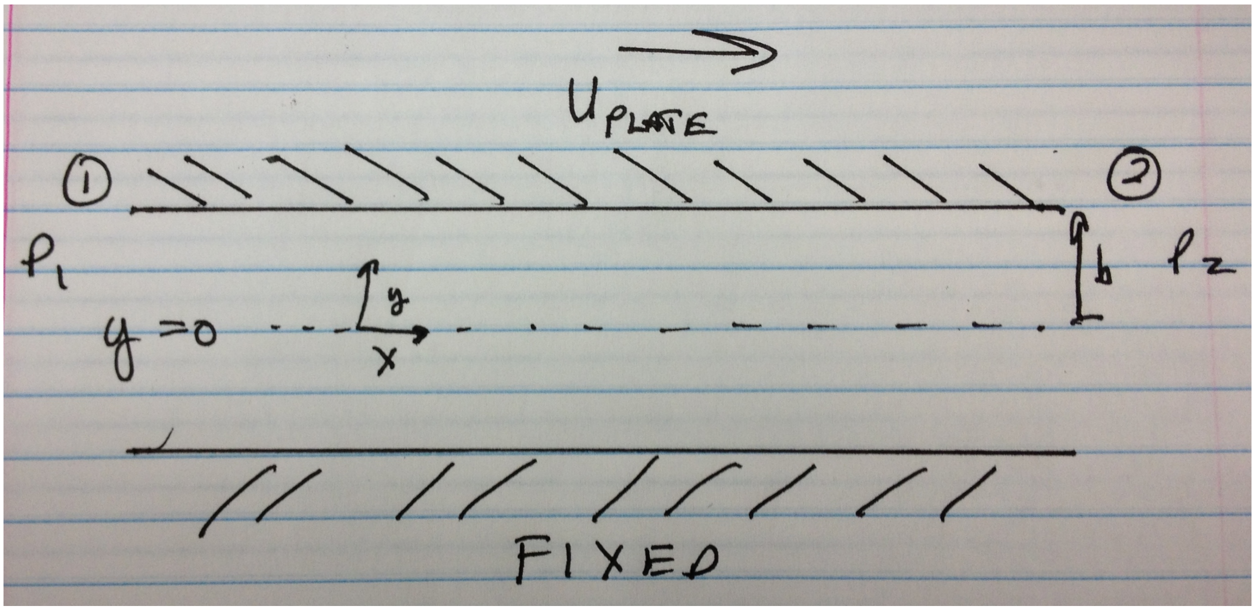
CBE 60544
Spring 2014
Final Exam
5/8/14

1. **Steady and startup flow between parallel plates**



Consider a Newtonian liquid with viscosity, μ and density ρ, flowing between parallel plates of infinite extent. We will analyze steady and transient flows caused by the top plate moving and the two pressures, P_1 and P_2 being unequal. Note that gravity does not appear in this problem!

a. For steady flows, is the combined flow caused by a moving top wall and a pressure imbalance the sum of the separately-derived solution for a moving top wall with equal P's and the fixed top wall and unequal P's? Explain why or why not with reference to appropriate mathematics.
b. Solve the steady, separate problems and find velocity profile for each case.
c. Find the average velocity for each separate case.
d. For equal average velocity, with the same plate spacing, find the ratio of the power required to cause the steady flow.
e. Explain *why* which ever flow configuration is more efficient, is more efficient.
f. Now <u>solve for the velocity profile</u> as a function of time for the <u>moving plate problem (no pressure gradient!)</u> with an initial state of zero velocity everywhere. Assume that at t = 0, the top plate is suddenly set to a velocity of Up.
g. How "long" does it take to reach a steady state? Give a mathematical relation from your results.
h. Find a relation for the <u>cumulative work done to start up</u> the flow all the way up to a steady flow.

2. **Lubrication flow in a knee joint**

The adjacent photo[1] shows a knee joint, apparently under load, with a spacing between the imaged bones of 4.5-6.3 mm. Presumably, much of this space is filled with cartilage with the remaining gap filled with synovial fluid.

The intent of this problem is to examine two of the "model" flows that are associated with load bearing joints, the "slider" and the "squeeze film". If we solve these two cases and enter some numbers we can find out that no matter how we might view the flow geometry and the mechanism of generation of the load bearing, the real gap will need to be considerably less that 5 mm.

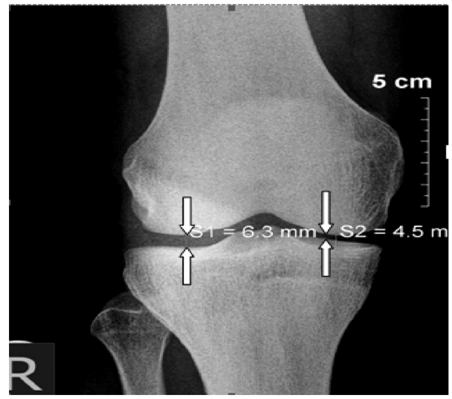


Fig. 2 The frontal 30° fixed-flexion weight-bearing knee computed radiograph; showing the reference points for measuring medial and lateral tibio-femoral joint space widths (arrows).

<u>Slider Flow.</u> We can use the figure from the text for this.

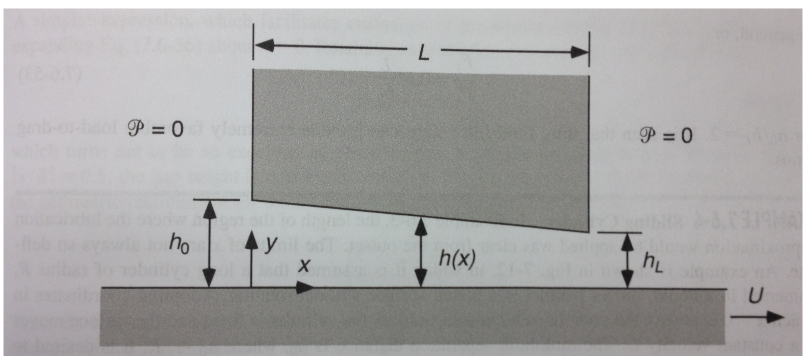


Figure 7-11. Slider bearing.

a. Find the velocity profile within the gap for the combined pressure driven/moving surface flow using the "nearly-parallel" assumption. Note that this means that locally (at any x) you can assume that the profile is a fully-developed flow with a parallel plate geometry. It also means that even if the outside pressure is uniform, there is a pressure gradient within the gap in (only) the x direction.
b. If you realize that whatever "*q*", the volumetric flow is, it does not change with x and if you solve for the pressure gradient and then integrate the result for the length of the gap, you will have a relationship between the pressure change (which could now be set to 0), *U*, the geometry and "*q*", the volumetric flow. Do this and find "*q*" in terms of U, μ and the geometric variables.
c. Now find the pressure as a function of *x* along the gap.
d. Calculate the total "load" that the slider could support.

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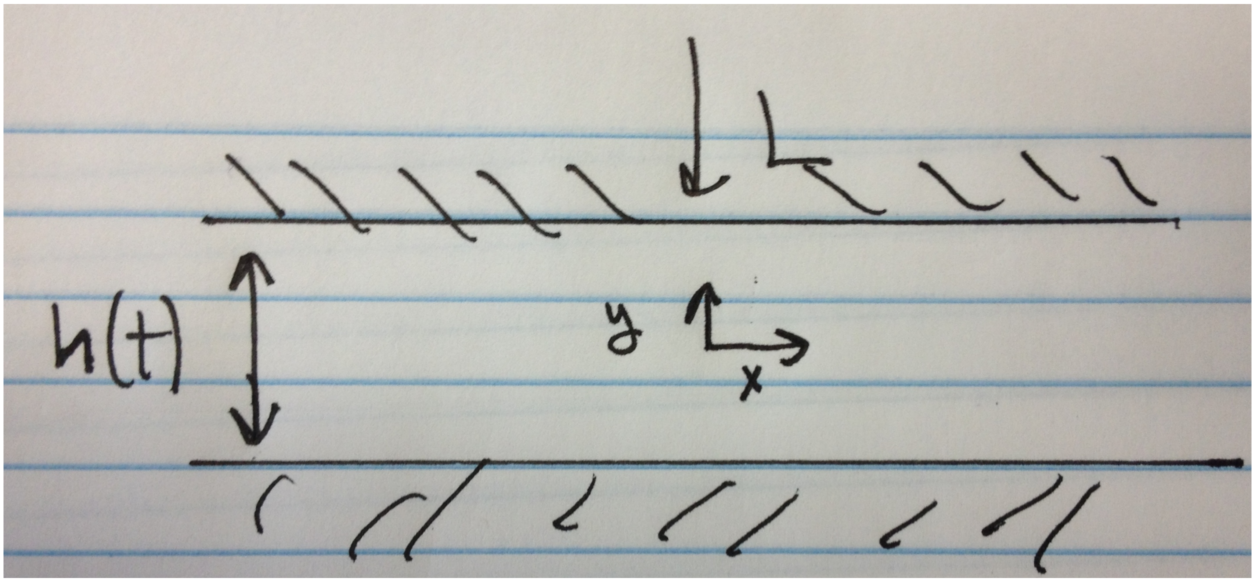
[1] Anas et al. (2013) Egyptian Journal of Radiology and Nuclear Medicine **44** 253-258.

e. If the h_0 = 0.6 cm, h_L = 0.4 cm and L = 10 cm (with a width of 4 cm), for a fluid viscosity of 2 Poise (i.e. g/cm-s), what is total load that could be supported? Is this anything close to what your knee must be supporting.

Squeeze film



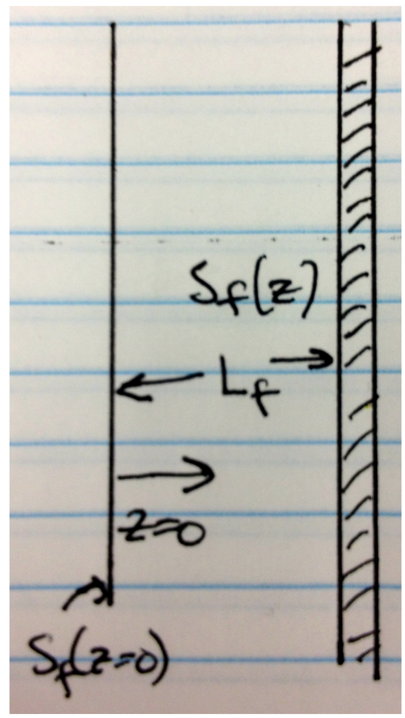
Consider the knee joint as modeled by a "squeeze" film. That is, when a load is placed on this geometry, there is a transient where fluid is being squeezed from the gap. You can assume that except for a short distance near x=0, that the flow is pressure driven and fully-developed with a pseudo-steady state such that H = h(t). That is, the flow instantaneously adjusts to the gap spacing.

f. Find an expression for the relationship between the load that can be supported and the rate of change of the plate spacing.

g. From your solution or by other reasoning, explain why the greatest load can be supported, for a given dh/dt, when the gap is smallest.

3. Analysis of a "biofilm"

While a bacterium could exist, at least for some amount of time, as a discrete, isolated entity, if a solid surface is available bacteria will be found together in large numbers in organized structures that can be termed "biofilms". If you leave a glass of water out for several hours you can grow your very own biofilm on the wall of the glass. When you rinse it out, the substance on the wall that is more viscous than water is a mixture of bacteria and various molecules secreted by the bacteria to make an structure that aids their survival. In particular, a biofilm that forms on an intravenous needle or tube or on an implanted medical device could enable bacteria growth and resistance to antibiotic treatment.



As a first level analysis, let's consider a biofilm that is at steady state. To achieve this bacteria will need a source of nutrients that is diffusing in from the adjacent water. If we consider "S_f", as the concentration of the rate limiting "substrate" in the film, the equation for steady state concentration of S_f will be[2]

$$0 = D_f \frac{\partial^2 S_f}{\partial z^2} - \frac{k X_f S_f}{K_s + S_f}$$

where we have assumed "Monod" or Michaelis-Menten or Langmuir-Hinshelwood (depending on your lingo preference) kinetics. In this equation D_f is the diffusivity, k is simple rate constant for the consumption of S, K_s is measure of the maximum amount of S that could be available for consumption by the bacteria within the "physiology" of the film.

a. If the substrate concentration in the liquid is S_0, which is also the interfacial concentration of $S_f(z=0)$ and the resulting concentration within the film is such that $S_f << K_s$, find an expression for the $S_f(z)$ within the film. The thickness of the biofilm is L_f.
b. What dimensionless parameter controls the behavior of this solution.
c. Sketch some concentration profiles as this parameter is varied.
d. Now consider the same problem in the limit of S_0 being a maximal value so that that S is in great excess. Find the profile $S_f(z)$ for this case.
e. Suppose that the biofilm is actively consuming S and the water next to the film is stagnant, for some depth L_w, draw a new "picture" of the mass transfer process, write down the requisite transport equations and boundary equations. There is no need to solve these.

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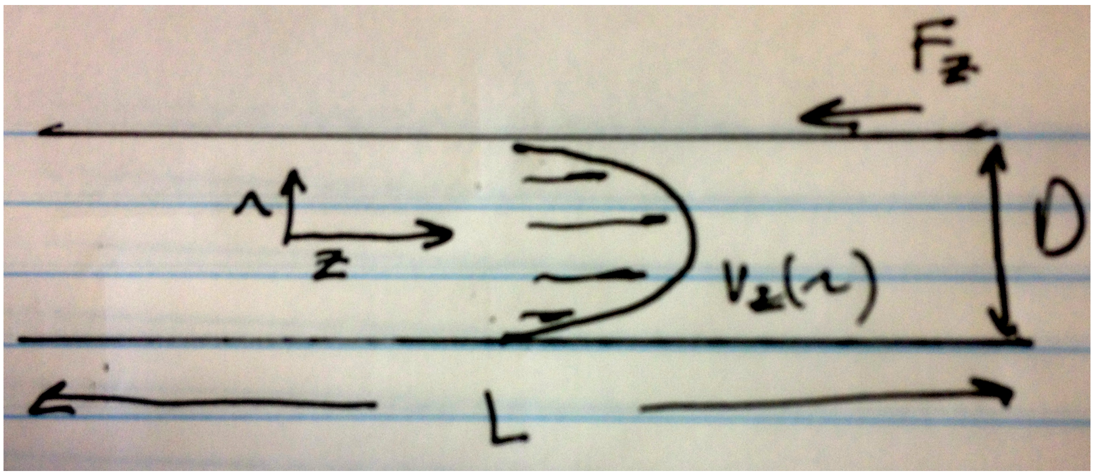
[2] Rittmann and McCarty (1980) Biotechnology and Bioengineering **22** pp2343-2357

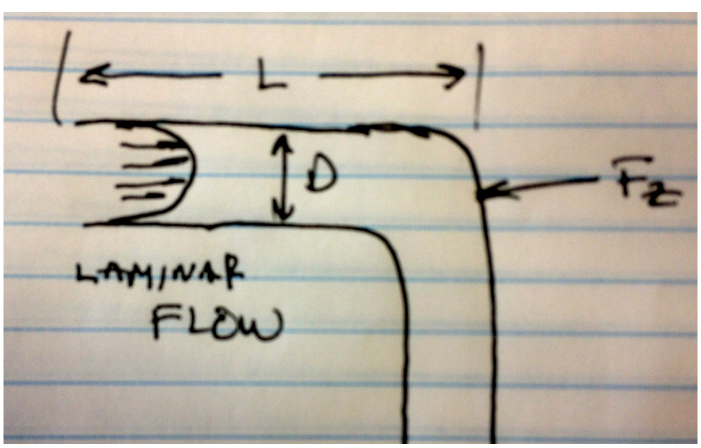
4. Short answers.

a. Explain why “stir faster” might not be as good a “use a hammer” for enhancing the rate of dissolution of a particulate solid in a solvent.

b. For a Newtonian fluid in a circular pipe, find an expression for F_z, the force on the outside of the pipe necessary to keep the pipe in place.



c. For a Newtonian fluid in a circular pipe, find an expression for F_z, the force on the outside of the pipe necessary to keep the pipe in place.



$$\frac{\partial C_i}{\partial t} = -\nabla \cdot \mathbf{N}_i + R_{Vi}.$$

$$= C_i\mathbf{v} + \mathbf{J}_i = C_i\mathbf{v}^{(M)} + \mathbf{J}_i^{(M)}, \qquad \sum_{i=1}^{n} \mathbf{N}_i = C\mathbf{v}^{(M)}, \qquad \sum_{i=1}^{n} \mathbf{J}_i^{(M)} = \mathbf{0} \qquad (1.2\text{-}8)$$

Table 1-3

Fick's Law for Binary Mixtures of *A* and *B*

Reference velocity	Mass units	
$\mathbf{v}$	$\mathbf{j}_A = -\rho D_{AB}\nabla\omega_A$	(A)
$\mathbf{v}^{(M)}$	$\mathbf{j}_A^{(M)} = -CM_A D_{AB}\nabla x_A$	(C)

Table 1-2

Flux of Species *i* in Various Reference Frames and Units

Reference velocity	Molar units	Mass units
$\mathbf{0}$	$\mathbf{N}_i$	$\mathbf{n}_i$
$\mathbf{v}$	$\mathbf{J}_i$	$\mathbf{j}_i$
$\mathbf{v}^{(M)}$	$\mathbf{J}_i^{(M)}$	$\mathbf{j}_i^{(M)}$

TABLE 2-2

Continuity Equation in Rectangular, Cylindrical, and Spherical Coordinates

Rectangular (x, y, z, t)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical (r, θ, z, t)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical (r, θ, ϕ, t)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

ABLE 2-4

Species Conservation Equations for a Binary or Pseudobinary Mixture in Rectangular, Cylindrical, and Spherical Coordinates[a]

Rectangular (x, y, z, t)

$$\frac{\partial C_i}{\partial t} + v_x\frac{\partial C_i}{\partial x} + v_y\frac{\partial C_i}{\partial y} + v_z\frac{\partial C_i}{\partial z} = D_i\left[\frac{\partial^2 C_i}{\partial x^2} + \frac{\partial^2 C_i}{\partial y^2} + \frac{\partial^2 C_i}{\partial z^2}\right] + R_{Vi}$$

Cylindrical (r, θ, z, t)

$$\frac{\partial C_i}{\partial t} + v_r\frac{\partial C_i}{\partial r} + \frac{v_\theta}{r}\frac{\partial C_i}{\partial \theta} + v_z\frac{\partial C_i}{\partial z} = D_i\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_i}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 C_i}{\partial \theta^2} + \frac{\partial^2 C_i}{\partial z^2}\right] + R_{Vi}$$

Spherical (r, θ, ϕ, t)

$$\frac{\partial C_i}{\partial t} + v_r\frac{\partial C_i}{\partial r} + \frac{v_\theta}{r}\frac{\partial C_i}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial C_i}{\partial \phi} = D_i\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C_i}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial C_i}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 C_i}{\partial \phi^2}\right] + R_{Vi}$$

[a] It is assumed that ρ and D_i are constant, where D_i is the binary or pseudobinary diffusivity.

LE 2-3

pproximate Forms of the Energy Conservation Equation in Rectangular, herical Coordinates (Thermal Effects Only)[a]

ctangular (x, y, z, t)

$$\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{H_V}{\rho \hat{C}_p}$$

lindrical (r, θ, z, t)

$$\begin{aligned}
\langle \mathcal{L}_x\Theta, \Phi_n\rangle &= \int_a^b \frac{1}{w}\left[\frac{\partial}{\partial x}\left(p\frac{\partial \Theta}{\partial x}\right) + q\Theta\right]\Phi_n w\,dx \\
&= \int_a^b \frac{\partial}{\partial x}\left(p\frac{\partial \Theta}{\partial x}\right)\Phi_n dx + \int_a^b q\Theta\Phi_n\,dx \\
&= p\left(\frac{\partial \Theta}{\partial x}\Phi_n - \Theta\frac{d\Phi_n}{dx}\right)\Bigg|_{x=a}^{x=b} + \int_a^b \Theta\left[\frac{d}{dx}\left(p\frac{d\Phi_n}{dx}\right) + q\Phi_n\right]dx \\
&= p\left(\frac{\partial \Theta}{\partial x}\Phi_n - \Theta\frac{d\Phi_n}{dx}\right)\Bigg|_{x=a}^{x=b} - \lambda_n^2\Theta_n.
\end{aligned} \tag{5.4-1}$$

Table 5-1
Original and Subsidiary Problems in a Linear Superposition

Equation	$\Theta(x,t)$	$f(x)$	$\psi(x,t)$
DE	$\frac{\partial \Theta}{\partial t}=\frac{\partial^2 \Theta}{\partial x^2}+1$	$\frac{d^2 f}{dx^2}=-1$	$\frac{\partial \psi}{\partial t}=\frac{\partial^2 \psi}{\partial x^2}$
BC at $x=0$	$\Theta(0,t)=0$	$f(0)=0$	$\psi(0,t)=0$
BC at $x=1$	$\Theta(1,t)=0$	$f(1)=0$	$\psi(1,t)=0$
IC	$\Theta(x,0)=0$	NA	$\psi(x,0)=-f(x$

Abbreviations: DE, differential equation; BC, boundary condition; IC, initial condition; NA, not applicable.

Orthonormal Functions from Eigenvalue Problems in Cartesian Coordinates[a]

Case	Boundary conditions	Basis functions
I	$\Phi(0)=0=\Phi(\ell)$	$\Phi_n(x)=\sqrt{\frac{2}{\ell}}\sin\frac{n\pi x}{\ell},\ n=1,2,\ldots$
II	$\frac{d\Phi}{dx}(0)=0=\Phi(\ell)$	$\Phi_n(x)=\sqrt{\frac{2}{\ell}}\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{\ell},\ n=0,1,2,\ldots$
III	$\Phi(0)=0=\frac{d\Phi}{dx}(\ell)$	$\Phi_n(x)=\sqrt{\frac{2}{\ell}}\sin\left(n+\frac{1}{2}\right)\frac{\pi x}{\ell},\ n=0,1,2,\ldots$
IV	$\frac{d\Phi}{dx}(0)=0=\frac{d\Phi}{dx}(\ell)$	$\Phi_n(x)=\sqrt{\frac{2}{\ell}}\cos\frac{n\pi x}{\ell},\ n=1,2,\ldots$
		$\Phi_0(x)=\frac{1}{\sqrt{\ell}}$

[a]All of these functions satisfy $d^2\Phi/dx^2=-\lambda^2\Phi$ for $0\le x\le\ell$.

Table 5-3
FFT Problems and Corresponding Eigenvalue Problems

	Geometry	Partial differential equation	Coord.	w	p	Eigenfunctions
1	Rectangular (x,t)	$\frac{\partial\Theta}{\partial t}=\frac{\partial^2\Theta}{\partial x^2}$	x	1	1	$\sin\lambda x,\ \cos\lambda x$
2	Rectangular (x,y)	$\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}=0$	x	1	1	$\sin\lambda x,\ \cos\lambda x$
			y	1	1	$\sin\lambda y,\ \cos\lambda y$
3	Cylindrical (r,t)	$\frac{\partial\Theta}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right)$	r	r	r	$J_0(\lambda r),\ Y_0(\lambda r)$
4	Cylindrical (r,z)	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right)+\frac{\partial^2\Theta}{\partial z^2}=0$	r	r	r	$J_0(\lambda r),\ Y_0(\lambda r)$
			z	1	1	$\sin\lambda z,\ \cos\lambda z$
5	Cylindrical (r,θ)	$r\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right)+\frac{\partial^2\Theta}{\partial\theta^2}=0$	θ	1	1	$\sin\lambda\theta,\ \cos\lambda\theta$
6	Spherical (r,t)	$\frac{\partial\Theta}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Theta}{\partial r}\right)$	r	r^2	r^2	$\frac{\sin\lambda r}{r},\ \frac{\cos\lambda r}{r}$
7	Spherical $(r,\eta=\cos\theta)$	$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Theta}{\partial r}\right)+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Theta}{\partial\eta}\right]=0$	η	1	$1-\eta^2$	$P_n(\eta)$

```
DSolve[{ D[c[y], {y, 2}] - ϕ^2 c[y] == 0}, c[y], y]
```

Out[185]= $\{\{c[y] \to e^{y\phi} C[1] + e^{-y\phi} C[2]\}\}$

```
In[186]:= ExpToTrig[ans]
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```
{{c[y] → C[3] Cosh[y ϕ] + C[4] Sinh[y ϕ]}}
```

Multiplying P by the identity tensor ensures that pressure is a normal stress (con only to the diagonal elements of $\boldsymbol{\sigma}$) and is isotropic (gives diagonal elements that are The minus sign is needed to make positive pressures compressive.

In a fluid undergoing any type of deformation, the total stress is writte generally as

$$\boldsymbol{\sigma} = -P\boldsymbol{\delta} + \boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ is the *viscous stress tensor* (or *deviatoric stress*). In a fluid at rest, this exce vanishes.[1] The viscous stress is related to the rate of deformation of the fluid, as a in Sections 6.4 and 6.5. Because the off-diagonal components of $\boldsymbol{\sigma}$ equal those c symmetry of $\boldsymbol{\sigma}$ implies that $\boldsymbol{\tau}$ is also symmetric.

The pressure in Eqs. (6.3-7) and (6.3-8) is the same as in thermodynamics a obey an equation of state of the form $P = P(\rho, T)$, such as the ideal gas law. For mal flow of single-component fluids, the unknowns in general are $\mathbf{v}$, P, and ρ, and t erning equations are the continuity equation, conservation of momentum (incl constitutive equation for $\boldsymbol{\tau}$), and the equation of state. Incompressible fluids, the ma in this book, are an idealization in which ρ is a specified constant. This reduces the of unknowns by one, eliminates the equation of state, and makes P simply a mechani able that adjusts to satisfy continuity and conservation of momentum. Unless it is s at a boundary, the absolute value of P in an incompressible fluid is arbitrary. The r ship between P and the mean normal stress in a flowing fluid is discussed in Sectic

CAUCHY MOMENTUM EQUATION

Using Eq. (6.3-8), the divergence of the total stress is

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (-P\boldsymbol{\delta}) + \nabla \cdot \boldsymbol{\tau} = -\nabla P + \nabla \cdot \boldsymbol{\tau}.$$

Accordingly, Eq. (6.2-27) becomes

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P + \nabla \cdot \boldsymbol{\tau}$$

which is called the *Cauchy momentum equation.* This general statement of conservatio ear momentum provides the starting point for analyzing both Newtonian and non-Ne flows, for either constant or variable ρ. Its components in rectangular (Cartesian), cyli and spherical coordinates are given in Tables 6-1, 6-2, and 6-3, respectively.

TABLE 6-1
Cauchy Momentum Equation in Rectangular Coordinates

x component	$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial P}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$
y component	$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial P}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]$
z component	$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial P}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$

[1] Although $\boldsymbol{\tau}$ is called the viscous stress tensor (in keeping with the applications in this book), there m elastic contributions to the nonequilibrium part of the total stress. For fluids that undergo structural r ments with noticeable relaxation times, "fluid at rest" carries with it the understanding that enough elapsed to reach mechanical equilibrium.

TABLE 6-2
Cauchy Momentum Equation in Cylindrical Coordinates

r component	$\rho\left[\dfrac{\partial v_r}{\partial t} + v_r\dfrac{\partial v_r}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta^2}{r} + v_z\dfrac{\partial v_r}{\partial z}\right]$ $= \rho g_r - \dfrac{\partial P}{\partial r} + \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rr}) + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} - \dfrac{\tau_{\theta\theta}}{r} + \dfrac{\partial \tau_{zr}}{\partial z}\right]$
θ component	$\rho\left[\dfrac{\partial v_\theta}{\partial t} + v_r\dfrac{\partial v_\theta}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r v_\theta}{r} + v_z\dfrac{\partial v_\theta}{\partial z}\right]$ $= \rho g_\theta - \dfrac{1}{r}\dfrac{\partial P}{\partial \theta} + \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\tau_{r\theta}) + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + \dfrac{\partial \tau_{z\theta}}{\partial z}\right]$
z component	$\rho\left[\dfrac{\partial v_z}{\partial t} + v_r\dfrac{\partial v_z}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial v_z}{\partial \theta} + v_z\dfrac{\partial v_z}{\partial z}\right]$ $= \rho g_z - \dfrac{\partial P}{\partial z} + \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rz}) + \dfrac{1}{r}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \tau_{zz}}{\partial z}\right]$

STATIC AND DYNAMIC PRESSURES

For $\mathbf{v} = \mathbf{0}$, Eq. (6.3-10) reduces to the *static pressure equation*,

$$\nabla P = \rho \mathbf{g}. \tag{6.3-11}$$

This encapsulates the well-known fact that the pressure in a static fluid increases with depth (i.e., in the direction of $\mathbf{g}$). If ρ is constant and the z axis points upward, then $\mathbf{g} = -g\mathbf{e}_z$, $dP/dz = -\rho g$, and

$$P(z) = P(0) - \rho g z. \tag{6.3-12}$$

In terms of a position vector, the pressure in a constant-density fluid at rest is given by

$$P(\mathbf{r}) = P(\mathbf{0}) + \rho \mathbf{g}\cdot\mathbf{r} \tag{6.3-13}$$

where $P(\mathbf{0})$ is the pressure at the origin. The term $\rho\mathbf{g}\cdot\mathbf{r}$ is equivalent to $-\rho g h$, where h is height above the origin.

TABLE 6-3
Cauchy Momentum Equation in Spherical Coordinates

r component	$\rho\left[\dfrac{\partial v_r}{\partial t} + v_r\dfrac{\partial v_r}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial v_r}{\partial \theta} + \dfrac{v_\phi}{r\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{v_\theta^2 + v_\phi^2}{r}\right]$ $= \rho g_r - \dfrac{\partial P}{\partial r} + \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\tau_{rr}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\tau_{\theta r}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial \tau_{\phi r}}{\partial \phi} - \dfrac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right]$
θ component	$\rho\left[\dfrac{\partial v_\theta}{\partial t} + v_r\dfrac{\partial v_\theta}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_\phi}{r\sin\theta}\dfrac{\partial v_\theta}{\partial \phi} + \dfrac{v_r v_\theta}{r} - \dfrac{v_\phi^2\cot\theta}{r}\right]$ $= \rho g_\theta - \dfrac{1}{r}\dfrac{\partial P}{\partial \theta} + \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\tau_{r\theta}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial \tau_{\phi\theta}}{\partial \phi} + \dfrac{\tau_{r\theta}}{r} - \dfrac{\cot\theta}{r}\tau_{\phi\phi}\right]$
ϕ component	$\rho\left[\dfrac{\partial v_\phi}{\partial t} + v_r\dfrac{\partial v_\phi}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial v_\phi}{\partial \theta} + \dfrac{v_\phi}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{v_\phi v_r}{r} + \dfrac{v_\theta v_\phi\cot\theta}{r}\right]$ $= \rho g_\phi - \dfrac{1}{r\sin\theta}\dfrac{\partial P}{\partial \phi} + \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\tau_{r\phi}) + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\phi}}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial \tau_{\phi\phi}}{\partial \phi} + \dfrac{\tau_{r\phi}}{r} + \dfrac{2\cot\theta}{r}\tau_{\theta\phi}\right]$

to suppose that the molecular-level resistance to dilatation and to shape-changing deformations might differ, leading to different proportionality constants for the kinds of motion described by Eqs. (6.4-15) and (6.4-16). This yields the result for a *Newtonian fluid*,

$$\boldsymbol{\tau} = 2\mu\left[\boldsymbol{\Gamma} - \frac{1}{3}(\nabla\cdot\mathbf{v})\boldsymbol{\delta}\right] + 3\kappa\left[\frac{1}{3}(\nabla\cdot\mathbf{v})\boldsymbol{\delta}\right]$$
$$= 2\mu\boldsymbol{\Gamma} + \left(\kappa - \frac{2}{3}\mu\right)(\nabla\cdot\mathbf{v})\boldsymbol{\delta}. \qquad (6.5\text{-}1)$$

As shown in the first line, the convention is to write the coefficients of the respective rate-of-strain tensors as 2μ and 3κ, where μ is the shear viscosity discussed in Chapter 1 and κ is the *dilatational viscosity* or *bulk viscosity*. Although the coefficients depend on composition and may also vary with T and P, the defining feature of a Newtonian fluid is that μ and κ do not depend on the magnitudes of $\boldsymbol{\Gamma}$ or $\nabla\cdot\mathbf{v}$.

The effects of κ on fluid dynamics are difficult to detect and are usually ignored. It has been shown theoretically that $\kappa = 0$ for ideal monatomic gases, and it is generally believed that for other fluids $\kappa \ll \mu$. Contributing to the difficulty in measuring κ is that $\nabla\cdot\mathbf{v} = 0$ when the density is constant. Constant ρ is a good approximation for liquids and even for many gas flows, and in the absence of dilatation κ cannot affect the viscous stress. For these reasons it is assumed hereafter that $\kappa = 0$. An analysis involving a dilatational viscosity is given in Batchelor (1970, pp. 253–55), in which a liquid containing gas bubbles is modeled as a homogeneous fluid.

Setting $\kappa = 0$ in Eq. (6.5-1), the constitutive equation for a Newtonian fluid reduces to

$$\boldsymbol{\tau} = 2\mu\left[\boldsymbol{\Gamma} - \frac{1}{3}(\nabla\cdot\mathbf{v})\boldsymbol{\delta}\right]. \qquad (6.5\text{-}2)$$

This is given in component form for the three common coordinate systems in Tables 6-4, 6-5, and 6-6. For an *incompressible* Newtonian fluid (constant ρ), the viscous stress is

$$\boldsymbol{\tau} = 2\mu\boldsymbol{\Gamma} = \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^t] \qquad (6.5\text{-}3)$$

TABLE 6-4
Viscous Stress Components for Newtonian Fluids in Rectangular Coordinates

$$\tau_{xx} = 2\mu\left[\frac{\partial v_x}{\partial x} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{yy} = 2\mu\left[\frac{\partial v_y}{\partial y} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{zz} = 2\mu\left[\frac{\partial v_z}{\partial z} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right]$$

$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right]$$

$$\tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right]$$

$$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

TABLE 6-5
Viscous Stress Components for Newtonian Fluids in Cylindrical Coordinates

$$\tau_{rr} = 2\mu\left[\frac{\partial v_r}{\partial r} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{\theta\theta} = 2\mu\left[\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{zz} = 2\mu\left[\frac{\partial v_z}{\partial z} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

as stated in Chapter 1. For a unidirectional flow with $v_x = v_x(y)$ and $v_y = v_z = 0$, it may be confirmed using Table 6-4 that Eq. (6.5-3) reduces to Eq. (1.2-14).

In Section 6.3 the total stress was related to the pressure and viscous stress by

$$\boldsymbol{\sigma} = -P\boldsymbol{\delta} + \boldsymbol{\tau} \tag{6.5-4}$$

where P was equated with the thermodynamic pressure. An alternative, purely mechanical definition of pressure is that it is (minus) the *mean normal stress* (Batchelor, 1970, p. 141; Panton, 1996, pp. 97–99). If this alternative pressure variable is denoted as $\overline{P}$, then

$$\overline{P} \equiv -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \tag{6.5-5}$$

TABLE 6-6
Viscous Stress Components for Newtonian Fluids in Spherical Coordinates

$$\tau_{rr} = 2\mu\left[\frac{\partial v_r}{\partial r} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{\theta\theta} = 2\mu\left[\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{\phi\phi} = 2\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r} - \frac{1}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Using Eqs. (6.5-1) and (6.5-4) to evaluate σ_{xx} for a Newtonian fluid gives

$$\sigma_{xx} = -P + 2\mu \frac{\partial v_x}{\partial x} + \left(\kappa - \frac{2}{3}\mu\right)(\nabla \cdot \mathbf{v}). \tag{6.5-6}$$

Substitution of this and the analogous expressions for σ_{yy} and σ_{zz} in Eq. (6.5-5) results in

$$\overline{P} = P - \kappa(\nabla \cdot \mathbf{v}). \tag{6.5-7}$$

Thus, P and $\overline{P}$ are identical in any Newtonian fluid that is static, incompressible, or has $\kappa = 0$.

NAVIER–STOKES EQUATION

The most widely encountered special case is a Newtonian fluid of constant density and viscosity. Using Eqs. (6.5-3) and (A.4-11), it is found that

$$\nabla \cdot \boldsymbol{\tau} = \mu[\nabla \cdot (\nabla \mathbf{v} + (\nabla \mathbf{v})^t)] = \mu[\nabla^2 \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})] = \mu \nabla^2 \mathbf{v}. \tag{6.5-8}$$

Substituting this result into Eq. (6.3-9) leads to

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P + \mu \nabla^2 \mathbf{v} \tag{6.5-9}$$

which is the *Navier–Stokes equation*.[4] Written in terms of the dynamic pressure, it is

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v}. \tag{6.5-10}$$

Either form of the Navier–Stokes equation, together with the continuity equation reduced to

$$\nabla \cdot \mathbf{v} = 0 \tag{6.5-11}$$

provides the usual starting point for analyzing the flow of simple liquids or gases at moderate velocities. As mentioned in Section 6.3, the unknowns for a pure, incompressible, isothermal fluid are just $\mathbf{v}$ and P. Counting the three scalar components of $\mathbf{v}$, there are a total of four unknown functions. The continuity equation plus the three components of the Navier–Stokes equation provide the requisite four partial differential equations. Equation (6.5-9) is given in component form for the common coordinate systems in Tables 6-7, 6-8, and 6-9, and $\nabla \cdot \mathbf{v}$ may be found in Tables 6-4, 6-5, and 6-6, as well as in Table 2-2.

TABLE 6-7
Navier–Stokes Equation in Rectangular Coordinates

x component	$\rho\left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right] = \rho g_x - \frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]$
y component	$\rho\left[\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right] = \rho g_y - \frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right]$
z component	$\rho\left[\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial P}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$

[4] This equation was reported independently by the French physicist L. Navier (1785–1836) in 1822 and the Anglo-Irish physicist G. G. Stokes (1819–1903) in 1845. A discussion of Navier's contributions is in Dugas (1988, pp. 409–414); additional information on Stokes is given in Chapter 8.

TABLE 6-8
Navier–Stokes Equation in Cylindrical Coordinates

r component
$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right]$$
$$= \rho g_r - \frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ component
$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right]$$
$$= \rho g_\theta - \frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z component
$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right]$$
$$= \rho g_z - \frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

NON-NEWTONIAN FLUIDS

Any fluid that does not obey Eq. (6.5-1) is non-Newtonian. In general, such fluids have an internal structure that is influenced by the flow and that in turn influences the relationship between the viscous stress and the rate of strain. Examples include polymer melts, concentrated polymer solutions, and suspensions of nonspherical and/or strongly interacting particles. Among the distinguishing characteristics of various non-Newtonian fluids are a dependence of the apparent viscosity on the rate of strain, unusually large values of the normal components of the viscous stresses, and "memory" effects in the relationship between the stress and the rate of strain. A dependence of viscosity on rate of strain may reflect the

TABLE 6-9
Navier–Stokes Equation in Spherical Coordinates

r component
$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right]$$
$$= \rho g_r - \frac{\partial P}{\partial r} + \mu\left[\nabla^2 v_r - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]$$

θ component
$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right]$$
$$= \rho g_\theta - \frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right]$$

ϕ component
$$\rho\left[\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi\cot\theta}{r}\right]$$
$$= \rho g_\phi - \frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \mu\left[\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

Laplacian
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial \phi^2}\right)$$

ability of the flow to orient particles or macromolecules, to break up particle aggregates, and/or to influence the conformation of polymer molecules. The resistance of long-chain polymers to orientation or elongation gives polymeric fluids an elastic character, which is manifested as an additional tension along streamlines (i.e., an elevated normal stress in the flow direction). The finite time required for such molecules to reach an equilibrium configuration is the source of memory effects. Fluids that exhibit elastic characteristics and have finite relaxation times are *viscoelastic*.

The dependence of the viscosity on the rate of strain is the major concern in many processes and can be modeled using straightforward modifications of the constitutive equation for a Newtonian fluid. The resulting equations describe *generalized Newtonian fluids*, which are discussed next. The analysis of viscoelastic phenomena, including normal stress and memory effects, is beyond the scope of this book. For discussions of that and other aspects of polymer rheology see Bird et al. (1987), Pearson (1985), or Tanner (1985). Bird et al. (1987) is the source for most of the information presented below on generalized Newtonian fluids.

GENERALIZED NEWTONIAN FLUIDS

In this discussion of generalized Newtonian fluids it is assumed that ρ is constant. A number of empirical expressions have been used to describe variations in the apparent viscosity with the rate of strain. Using Eq. (A.3-34) and recalling that $\mathbf{\Gamma}$ is symmetric, a scalar measure of the rate of strain is

$$\Gamma = \left[\frac{1}{2}(\mathbf{\Gamma}:\mathbf{\Gamma})\right]^{1/2}. \tag{6.5-12}$$

In a generalized Newtonian fluid the viscous stress is described still by Eq. (6.5-3), but now with $\mu = \mu(\Gamma)$. Expressions for Γ in the three common coordinate systems (valid even if ρ is not constant) are given in Table 6-10. For a flow in which $v_x = v_x(y)$ and $v_y = v_z = 0$, $\Gamma = |\Gamma_{yx}| = (1/2)|dv_x/d_y|$. The *shear rate* equals 2Γ, or $|dv_x/dy|$ in this example. Thus, each entry in Table 6-10 is the square of the shear rate.

TABLE 6-10
Magnitude of Rate of Strain

Rectangular	$(2\Gamma)^2 = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2$
Cylindrical	$(2\Gamma)^2 = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2$
Spherical	$(2\Gamma)^2 = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2$

TABLE 6-12
Stream Function Equations

Geometry	Velocity components	Form of Navier–Stokes equation[a]	Differential operators
Cartesian (x, y)	$v_x = \frac{\partial \psi}{\partial y}$ $v_y = -\frac{\partial \psi}{\partial x}$	$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = \nu\nabla^4\psi$	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $\nabla^4 = \nabla^2(\nabla^2)$
Cylindrical (r, θ)	$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ $v_\theta = -\frac{\partial \psi}{\partial r}$	$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{1}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r,\theta)} = \nu\nabla^4\psi$	$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ $\nabla^4 = \nabla^2(\nabla^2)$
Cylindrical (z, r)	$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$ $v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$	$\frac{\partial}{\partial t}(E^2\psi) + \frac{1}{r}\frac{\partial(\psi, E^2\psi)}{\partial(z,r)} + \frac{2}{r^2}\frac{\partial \psi}{\partial z}E^2\psi$ $= \nu E^4\psi$	$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ $E^4 = E^2(E^2)$
Spherical (r, θ)	$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$ $v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$	$\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r^2 \sin\theta}\frac{\partial(\psi, E^2\psi)}{\partial(r,\theta)}$ $+ \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial \psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\sin\theta\right)$ $= \nu E^4\psi$	$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)$ $E^4 = E^2(E^2)$

[a]The Jacobian determinants are given by

$$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix}.$$

Eq. (A.7-13) and identities in Table A-1] gives

$$\nabla \cdot \mathbf{v} = \nabla \cdot \left(\nabla\psi \times \frac{\mathbf{e}_3}{h_3}\right) = \frac{\mathbf{e}_3}{h_3} \cdot (\nabla \times \nabla\psi) - \nabla\psi \cdot \left(\nabla \times \frac{\mathbf{e}_3}{h_3}\right) = 0 \qquad (6.8\text{-}4)$$

which confirms that the continuity equation is satisfied.

The stream function can be applied to planar or axisymmetric problems. In terms of the three usual coordinate systems, *planar flows*, involving (x, y) or cylindrical (r, θ) coordinates, are ones that are independent of z and for which $v_z = 0$. *Axisymmetric flows*, involving cylindrical (r, z) or spherical (r, θ) coordinates, are independent of rotations about the z axis and have $v_\theta = 0$ or $v_\phi = 0$, respectively. The relationships between ψ and the velocity components for these four geometries are given in Table 6-12. For the axisymmetric cylindrical case the coordinates are arranged here as (z, r, θ), so that $\mathbf{e}_3 = \mathbf{e}_\theta$ and $h_3 = h_\theta = r$.

Rewriting the Navier–Stokes equation in terms of the stream function is facilitated by the relationships that exist between ψ and the vorticity vector. Using entry (6) of Table A-1 to evaluate the curl of Eq. (6.8-3) gives[5]

$$\mathbf{w} = \nabla \times \left(\nabla\psi \times \frac{\mathbf{e}_3}{h_3}\right) = -\nabla\psi \cdot \nabla\left(\frac{\mathbf{e}_3}{h_3}\right) + \nabla\psi\left(\nabla \cdot \frac{\mathbf{e}_3}{h_3}\right) - \frac{\mathbf{e}_3}{h_3}\nabla^2\psi. \qquad (6.8\text{-}5)$$

[5] There is also an $(\mathbf{e}_3/h_3) \cdot \nabla\nabla\psi$ term, but it vanishes because neither the planar nor the axisymmetric form of ∇ has an $\mathbf{e}_3$ component.

Using this in Eq. (10.6-15) gives

$$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} - \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \frac{DP}{Dt} + \boldsymbol{\tau} : \nabla \mathbf{v}. \tag{10.6-18}$$

This form of the energy equation is valid for any single-component fluid with a symmetric stress tensor. In comparison with Eq. (2.4-1) (without H_V), there are two additional terms on the right-hand side. The one involving the density and pressure is related to the work required to compress the fluid and will be referred to as the *compressibility* term. The one involving the viscous stress represents the conversion of kinetic energy to heat, caused by friction within the fluid. This irreversible process is called *viscous dissipation*.

For a *Newtonian fluid*, the rate of viscous dissipation is proportional to the viscosity and is evaluated as

$$\boldsymbol{\tau} : \nabla \mathbf{v} = \mu\left[(2\Gamma)^2 - \frac{2}{3}(\nabla \cdot \mathbf{v})^2\right] \equiv \mu \Phi. \tag{10.6-19}$$

The *viscous dissipation function*, Φ, is related to the shear rate and the divergence of the velocity, as shown. When 2Γ is calculated as in Table 6-10 and $\nabla \cdot \mathbf{v}$ is evaluated as in Tables 6-4 through 6-6, it is found that $\Phi \geq 0$. The fact that Φ cannot be negative is most evident for an incompressible fluid, for which $\Phi = (2\Gamma)^2$. Accordingly, viscous dissipation always acts as a heat *source*, reflecting the irreversibility of frictional losses. For Newtonian fluids, including those with variable density, Eq. (10.6-18) becomes

$$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} - \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \frac{DP}{Dt} + \mu \Phi. \tag{10.6-20}$$

The compressibility term has a simple form for ideal gases, where $(\partial \rho / \partial T)_P = -\rho / T$. Thus, the energy equation for an *ideal gas* is

$$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \frac{DP}{Dt} + \mu \Phi. \tag{10.6-21}$$

For a *Newtonian fluid* that is *incompressible* in the sense that $(\partial \rho / \partial T)_P = 0$, the energy equation simplifies further to

$$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \mu \Phi. \tag{10.6-22}$$

The principal forms of the energy equation derived here are summarized in Table 10-1. Many other forms of conservation of energy are given in Bird et al. (2002, pp. 340–341).

The different uses of "incompressible" may be confusing, especially in connection with gases. From the viewpoint of the continuity and momentum equations, gases often behave as if their density is constant. As discussed in Section 2.3, this is generally true for velocities much smaller than the speed of sound. Nonetheless, the equation of state for a gas shows that $(\partial \rho / \partial T)_P \neq 0$. Whether Eq. (10.6-21) can be approximated by Eq. (10.6-22) depends on the relative magnitudes of the pressure and temperature variations. In this sense, there is a distinction between fluid-dynamic and thermodynamic incompressibility.

Table 10-1
Conservation of Energy for a Pure Fluid in Terms of Temperature and Pressure

General	$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} - \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \frac{DP}{Dt} + \boldsymbol{\tau} : \nabla \mathbf{v}$	(A)
Newtonian[a]	$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} - \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \frac{DP}{Dt} + \mu \Phi$	(B)
Ideal gas	$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \frac{DP}{Dt} + \mu \Phi$	(C)
Incompressible Newtonian[b]	$\rho \hat{C}_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \mu \Phi$	(D)

[a] The dissipation function, Φ, is defined in Eq. (10.6-19).
[b] As used here, "incompressible" means that $(\partial \rho / \partial T)_P = 0$.

Viscous dissipation introduces a new dimensionless parameter. For steady flow of a Newtonian, incompressible fluid, Eq. (10.2-1) becomes

$$\mathrm{Pe}\, \tilde{\mathbf{v}} \cdot \tilde{\nabla} \Theta = \tilde{\nabla}^2 \Theta + \mathrm{Br}\, \tilde{\Phi} \tag{10.6-23}$$

$$\tilde{\Phi} = \left(\frac{L}{U}\right)^2 \Phi \tag{10.6-24}$$

$$\mathrm{Br} = \frac{\mu U^2}{k \Delta T} \tag{10.6-25}$$

where ΔT is the temperature scale used in defining Θ. The *Brinkman number*, Br, expresses the relative importance of viscous dissipation and heat conduction. (A related dimensionless group is the *Eckert number*, Ec = Br/Pr.) Reasoning as in Section 10.3, it is inferred that

$$\mathrm{Nu} = \mathrm{Nu}(\tilde{\mathbf{r}}_w, \mathrm{Re}, \mathrm{Pr}, \mathrm{Br}, \text{geometric ratios}) \tag{10.6-26}$$

for steady flow. The results in Sections 10.4 and 10.5 all correspond to $\mathrm{Br} \to 0$, in which case viscous dissipation is negligible. Viscous dissipation tends to be important mainly for polymeric liquids (large μ) or high-speed flows (large Φ).

10.7 TAYLOR DISPERSION

Suppose that a soluble substance is injected rapidly into a liquid flowing in a tube, momentarily creating a sharp peak in solute concentration. As the peak moves downstream it will broaden, and detectors placed at various positions to measure the cross-sectional average concentration in the passing fluid might record a series of Gaussian profiles like those in Fig. 4-2. The dispersion of solute that broadens the peaks resembles molecular diffusion, but it occurs even if the Péclet number is large enough to make axial diffusion negligible. Adding to the seeming paradox, increasing the molecular diffusivity may slow the peak broadening, rather than speed it. At high Pe, the dispersion actually stems not from axial diffusion, but from the combined effects of radial diffusion and a nonuniform axial velocity. This phenomenon, analyzed by G. I. Taylor in the early 1950s, is called *Taylor*

TABLE 11-2
Forms of the Average Nusselt Number for Laminar Flow with Pe >> 1

Case	Other phase	Re	Pr	$\overline{\mathrm{Nu}}$
1	Fluid or solid	>> 1	Any[a]	$B(\mathrm{Pr})\,\mathrm{Re}^{1/2}$
2	Fluid	Any[a]	Any[a]	$C\,\mathrm{Re}^{1/2}\mathrm{Pr}^{1/2}$
3	Solid	<< 1 or ~ 1	>> 1	$C\,\mathrm{Re}^{1/3}\mathrm{Pr}^{1/3}$
4	Solid	>> 1	>> 1	$C\,\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}$
5	Solid	>> 1	<< 1	$C\,\mathrm{Re}^{1/2}\mathrm{Pr}^{1/2}$

[a] This parameter is not restricted individually, but must be such that Pe = Re Pr >> 1.

will scale as $\tilde{y}$, which in the thermal boundary layer is $O(\mathrm{Re}^{-a}\mathrm{Pr}^{-b})$ [see Eq. (11.4-3)]. Equating these exponents with those in Eq. (11.4-7) leads to $a = b = 1/3$. This scaling was seen in the analysis of heat transfer in creeping flow past a solid sphere for Pe >> 1 (Example 11.2-2).

FLUID–SOLID INTERFACE FOR LARGE Re AND LARGE Pr

Again, the first nonzero term in the expansion for the velocity is that proportional to $\tilde{y}$. What differs from the previous case is that, with a momentum boundary layer, the velocity gradient at the surface is large. Specifically, laminar boundary-layer theory predicts that $\partial\tilde{v}_x/\partial\tilde{y} = O(\mathrm{Re}^{1/2})$ (Chapter 9). It follows that the velocity in the thermal boundary layer is $O(\mathrm{Re}^{1/2-a}\mathrm{Pr}^{-b})$ and that $a = 1/2$ and $b = 1/3$. This scaling was derived in Section 11.3 [see Eq. (11.3-23)].

FLUID–SOLID INTERFACE FOR LARGE Re AND SMALL Pr

In boundary-layer flow at small Pr the thermal boundary layer resides mainly in the momentum outer region, as discussed in Section 11.3. Thus, it is known directly that $\tilde{v}_x = O(1)$ in the thermal boundary layer and Eq. (11.4-8) is not needed. Coincidentally, the scaling is the same as for a fluid–fluid interface, or $a = b = 1/2$. This result has been given already as Eq. (11.3-16).

SUMMARY OF SCALING LAWS

The results for the average Nusselt number in the situations just discussed are summarized in Table 11-2. The analogous predictions for the average Sherwood number are in Table 11-3. Case 1 in each table, which is for any situation with large Re, follows jointly from

TABLE 11-3
Forms of the Average Sherwood Number for Laminar Flow with Pe >> 1

Case	Other phase	Re	Sc	$\overline{\mathrm{Sh}}$
1	Fluid or solid	>> 1	Any[a]	$B(\mathrm{Sc})\,\mathrm{Re}^{1/2}$
2	Fluid	Any[a]	Any[a]	$C\,\mathrm{Re}^{1/2}\mathrm{Sc}^{1/2}$
3	Solid	<< 1 or ~ 1	>> 1	$C\,\mathrm{Re}^{1/3}\mathrm{Sc}^{1/3}$
4	Solid	>> 1	>> 1	$C\,\mathrm{Re}^{1/2}\mathrm{Sc}^{1/3}$

[a] This parameter is not restricted individually, but must be such that Pe = Re Sc >> 1.

Table 3.3 Mass transfer† for simple situations

Fluid motion	Range of conditions	Equation	Ref.
1. Inside circular pipes	Re = 4000–60 000 Sc = 0.6–3000	$j_D = 0.023\ \mathrm{Re}^{-0.17}$ $\mathrm{Sh} = 0.023\ \mathrm{Re}^{0.83}\ \mathrm{Sc}^{1/3}$	41, 52
	Re = 10 000 – 400 000 Sc > 100	$j_D = 0.0149\ \mathrm{Re}^{-0.12}$ $\mathrm{Sh} = 0.0149\ \mathrm{Re}^{0.88}\ \mathrm{Sc}^{1/3}$	44
2. Unconfined flow parallel to flat plates‡	Transfer begins at leading edge $\mathrm{Re}_x < 50\ 000$	$j_D = 0.664\ \mathrm{Re}_x^{-0.5}$	32
	$\mathrm{Re}_x = 5 \times 10^5 – 3 \times 10^7$ Pr = 0.7–380	$\mathrm{Nu} = 0.037\ \mathrm{Re}_x^{0.8}\ \mathrm{Pr}_0^{0.43}\left(\frac{\mathrm{Pr}_0}{\mathrm{Pr}_i}\right)^{0.25}$	
	$\mathrm{Re}_x = 2 \times 10^4 – 5 \times 10^5$ Pr = 0.7–380	Between above and $\mathrm{Nu} = 0.0027\ \mathrm{Re}_x\ \mathrm{Pr}_0^{0.43}\left(\frac{\mathrm{Pr}_0}{\mathrm{Pr}_i}\right)^{0.25}$	65
3. Confined gas flow parallel to a flat plate in a duct	$\mathrm{Re}_e = 2600–22\ 000$	$j_D = 0.11\ \mathrm{Re}_e^{-0.29}$	46
4. Liquid film in wetted-wall tower, transfer between liquid and gas	$\frac{4\Gamma}{\mu} = 0–1200$, ripples suppressed	Eqs. (3.18)–(3.22)	
	$\frac{4\Gamma}{\mu} = 1300–8300$	$\mathrm{Sh} = (1.76 \times 10^{-5})\left(\frac{4\Gamma}{\mu}\right)^{1.506}\mathrm{Sc}^{0.5}$	20, 37

5.	Perpendicular to single cylinders	Re = 400–25 000; Sc = 0.6–2.6	$\frac{k_G P_t}{G_M} Sc^{0.56} = 0.281\ Re'^{0.4}$	5
		Re′ = 0.1–10^5; Pr = 0.7–1500	$Nu = (0.35 + 0.34\ Re'^{0.5} + 0.15\ Re'^{0.58})\ Pr^{0.3}$	16, 21, 42
6.	Past single spheres	Sc = 0.6–3200; $Re''\ Sc^{0.5}$ = 1.8–600 000	$Sh = Sh_0 + 0.347(Re''\ Sc^{0.5})^{0.62}$; $Sh_0 = \begin{cases} 2.0 + 0.569(Gr_D\ Sc)^{0.250} & Gr_D\ Sc < 10^8 \\ 2.0 + 0.0254(Gr_D\ Sc)^{0.333}\ Sc^{0.244} & Gr_D\ Sc > 10^8 \end{cases}$	55
7.	Through fixed beds of pellets§	Re″ = 90–4000; Sc = 0.6	$j_D = j_H = \frac{2.06}{\varepsilon} Re''^{-0.575}$	4, 23, 64
		Re″ = 5000–10 300; Sc = 0.6	$j_D = 0.95 j_H = \frac{20.4}{\varepsilon} Re''^{-0.815}$	
		Re″ = 0.0016–55; Sc = 168–70 600	$j_D = \frac{1.09}{\varepsilon} Re''^{-2/3}$	
		Re″ = 5–1500; Sc = 168–70 600	$j_D = \frac{0.250}{\varepsilon} Re''^{-0.31}$	

† Average mass-transfer coefficients throughout, for constant solute concentrations at the phase surface. Generally, fluid properties are evaluated at the average conditions between the phase surface and the bulk fluid. The heat-mass-transfer analogy is valid throughout.

‡ Mass-transfer data for this case scatter badly but are reasonably well represented by setting $j_D = j_H$.

§ For fixed beds, the relation between ε and d_p is $a = 6(1 - \varepsilon)/d_p$, where a is the specific solid surface, surface per volume of bed. For mixed sizes [58]

$$d_p = \frac{\sum_{i=1}^{n} n_i d_{pi}^3}{\sum_{i=1}^{n} n_i d_{pi}^2}$$

TABLE A-2
Differential Operations in Rectangular (Cartesian) Coordinates[a]

(1)	$\nabla f = \frac{\partial f}{\partial x}\mathbf{e_x} + \frac{\partial f}{\partial y}\mathbf{e_y} + \frac{\partial f}{\partial z}\mathbf{e_z}$
(2)	$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
(3)	$\nabla \times \mathbf{v} = \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right]\mathbf{e_x} + \left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right]\mathbf{e_y} + \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]\mathbf{e_z}$
(4)	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
(5)	$(\nabla \mathbf{v})_{xx} = \frac{\partial v_x}{\partial x}$
(6)	$(\nabla \mathbf{v})_{xy} = \frac{\partial v_y}{\partial x}$
(7)	$(\nabla \mathbf{v})_{xz} = \frac{\partial v_z}{\partial x}$
(8)	$(\nabla \mathbf{v})_{yx} = \frac{\partial v_x}{\partial y}$
(9)	$(\nabla \mathbf{v})_{yy} = \frac{\partial v_y}{\partial y}$
(10)	$(\nabla \mathbf{v})_{yz} = \frac{\partial v_z}{\partial y}$
(11)	$(\nabla \mathbf{v})_{zx} = \frac{\partial v_x}{\partial z}$
(12)	$(\nabla \mathbf{v})_{zy} = \frac{\partial v_y}{\partial z}$
(13)	$(\nabla \mathbf{v})_{zz} = \frac{\partial v_z}{\partial z}$

[a]In these relationships f is any differentiable scalar function and **v** is any differentiable vector function.

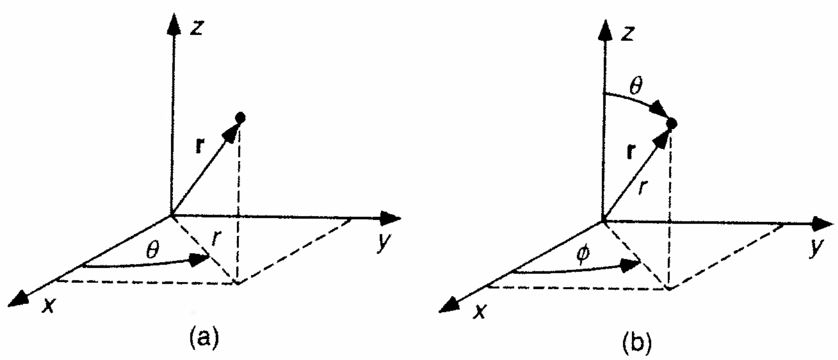

Figure A-3. Cylindrical coordinates (a) and spherical coordinates (b). The ranges of the angles are cylindrical, $0 \le \theta \le 2\pi$; and spherical, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.